Mathematical Methods study guide

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1 Einstein Notation

- Dot Product: $\{\overrightarrow{A} \cdot \overrightarrow{B}\} = A_i B_i$
- Inner Product: $\langle \overrightarrow{A}, \overrightarrow{B} \rangle = A_i \mu_{ij} B_j$
- Cross Product: $\{\overrightarrow{A} \times \overrightarrow{B}\}_i = \epsilon_{ijk} A_j B_k$
- Tensorial Product: $\{\overrightarrow{A} \bigotimes \overrightarrow{B}\} = A_i B_j$
- Kronecker's delta: $\delta_{ij} = \delta_{ji}$, $\delta_{ij} A_j = A_i$, $\delta_{ii} = 3$ (# dim)
- Levi Civita: $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$, $\epsilon_{ijk} = -\epsilon_{jik}$, $\epsilon_{iik} = 0$, $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$
- Derivatives: $\{\overrightarrow{r}\}_i = x_i$, $\{\nabla \phi\}_i = \frac{\partial \phi}{\partial x_i} = \partial_i \phi$

2 Derivatives and Integral Theorems

- Gradient: $\{\nabla \phi\}_i = \partial_i \phi$ it gives the rate of change and it points in the direction of the maximum increase.
- Divergence: $\nabla \cdot \overrightarrow{A} = \lim\limits_{\Delta\tau \to 0} \frac{flux\{\overrightarrow{A}, \Delta\tau\}}{\Delta\tau}$ where we define $flux\{\overrightarrow{A}, \tau\} = \oint\limits_{\mathcal{S}\, of\, \tau} \overrightarrow{A} \cdot \overrightarrow{d\sigma}$, it can be interpreted as the net flow out of the volume $d\tau$.
- Curl: $\lim\limits_{\Delta\overrightarrow{\sigma} \to \overrightarrow{0}} \Delta\overrightarrow{\sigma} \cdot (\nabla \times \overrightarrow{A}) = \lim\limits_{\Delta\overrightarrow{\sigma} \to \overrightarrow{0}} circ\{\overrightarrow{A}, \Delta\overrightarrow{\sigma}\}$ where we define $circ\{\overrightarrow{A}, \overrightarrow{\sigma}\} = \oint\limits_{\mathcal{C}\, of\, \overrightarrow{\sigma}} \overrightarrow{A} \cdot \overrightarrow{dl}$, and it can be interpreted as the circulation per unit of area.
- Fundamental theorem of gradients: if $\overrightarrow{F}(\overrightarrow{r}) = \nabla\phi(\overrightarrow{r})$ then $\int\limits_{\mathcal{C}\, \overrightarrow{r_a}}^{\overrightarrow{r_b}} \overrightarrow{F}(\overrightarrow{r}) \cdot \overrightarrow{dl} = \phi(\overrightarrow{r_b}) - \phi(\overrightarrow{r_a})$
- Gauss Theorem: if $F(\overrightarrow{r}) = \nabla \cdot \overrightarrow{A}(\overrightarrow{r})$ then $\int\limits_{\tau} \nabla \cdot \overrightarrow{A} d\tau = \int\limits_{\mathcal{S}} \overrightarrow{A} \cdot \overrightarrow{d\sigma}$
- Stokes Theorem: if $\overrightarrow{F}(\overrightarrow{r}) = \nabla \times \overrightarrow{A}(\overrightarrow{r})$ then $\int\limits_{\mathcal{S}} \nabla \times \overrightarrow{A} \cdot \overrightarrow{d\sigma} = \oint\limits_{\mathcal{C}} \overrightarrow{A} \cdot \overrightarrow{dl}$

3 Complex Variables and Functions

- Cauchy Riemann Conditions: $f(z) = u(x, y) + iv(x, y)$ is analytic if $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, a singularity is a point where they are not satisfied.
- Cauchy's integral theorem: if f is analytic inside and along $\mathcal{C}$ then $\oint\limits_{\mathcal{C}} f(z) dz = 0$

- <u>Cauchy's integral formula:</u> $f(z) = \frac{1}{2\pi i} \oint\limits_{\mathcal{C}} \frac{f(z')}{z'-z} dz'$

- <u>Taylor series:</u> $f(z) = \sum\limits_{n=0}^{\infty} f^{(n)}(z_o) \frac{(z-z_o)^n}{n!}$ distance between z and z_o must be smaller than the distance between z_o and the closest singularity.

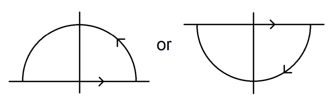
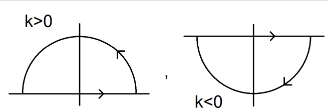
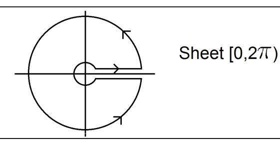
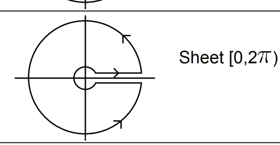
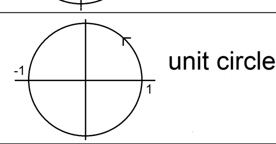
- <u>Laurent series:</u> $f(z) = \sum\limits_{n=-\infty}^{\infty} a_n (z-z_o)^n$ valid within an annular region centered at z_o, the singularities where we can expand the function as a Laurent series are called poles and if the series starts at the $-n$ power we call this singularity an n^{th} order pole.

- <u>Branch points, branch cuts & Riemann surfaces:</u> because of a branch point the function can take several values and we have # Riemann sheets = # of values of $f(z)$. A branch cut is an artificial line separating different Riemann sheets starting from the branch point. For the special case $f(z) = (z-z_o)^\alpha$ if $\alpha = \frac{M}{N}$ with no common prime factors then there are N Riemann sheets, for the extension $f(z) = (z-z_1)^\alpha (z-z_2)^\beta \dots$ we can choose multiple branch cuts, if the numbers $\alpha, \beta, \dots$ are rationals the number of Riemann sheets is the lowest common denominator and if the powers of a subset of branch points adds up to an integer then we can join them by a finite branch cut. If instead of a product we have a sum the number of Riemann sheets is the product of the number for each term.

- <u>Residue theorem:</u> $\oint\limits_{\mathcal{C}} f(z)dz = 2\pi i \sum\limits_{k\, inside\, \mathcal{C}} Res\{f(z); z_k\}$

- <u>Calculating residues:</u> for a n^{th} order pole $Res\{f(z); z_o\} = \lim\limits_{z \to z_o} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} [(z-z_o)^n f(z)]$ and for simple poles $Res\{\frac{g(z)}{h(z)}; z_o\} = \frac{g(z)}{h'(z)}$

Table 1: Solving definite integrals

Type	Use	Contour	Conditions
$I_1 = \int\limits_{-\infty}^{\infty} f(x)dx$	$\oint\limits_{\mathcal{C}} f(z)dz$		$f(z) \to 0$ faster than $\frac{1}{z}$ as $\lvert z\rvert \to \infty$
$I_2 = \int\limits_{-\infty}^{\infty} f(x)e^{ikx}dx$	$\oint\limits_{\mathcal{C}} f(z)e^{ikz}dz$		$f(z) \to 0$ as $\lvert z\rvert \to \infty$
$I_3 = \int\limits_{0}^{\infty} x^\alpha f(x)dx$	$\oint\limits_{\mathcal{C}} z^\alpha f(z)dz$		$z^\alpha f(z) \to 0$ faster than $\frac{1}{z}$ as $\lvert z\rvert \to \infty$
$I_4 = \int\limits_{0}^{\infty} f(x)dx$	$\oint\limits_{\mathcal{C}} \ln(z) f(z)dz$		$f(z) \to 0$ faster than $\frac{1}{z^{1+\delta}}$ with $\delta > 0$ as $\lvert z\rvert \to \infty$ and $f(0)$ is finite
$I_5 = \int\limits_{0}^{2\pi} F(\cos\theta, \sin\theta)d\theta$	$\oint\limits_{u.c} f(\frac{z+1/z}{2}, \frac{z-1/z}{2i}) \frac{dz}{iz}$		none

- <u>Infinite sums:</u> $\sum\limits_{n=-\infty}^{\infty} f(n) = -\sum\limits_{k, \text{poles of f}} \text{Res}\{\gamma(z)f(z); z_k\}$ where $\gamma(z) = \pi \cot(\pi z)$.

- Principal values: when we have simple poles along the contour,

$$\mathcal{P}\oint_{\mathcal{C}} f(z)dz = 2\pi i \sum_{k\, inside\, \mathcal{C}} Res\{f(z); z_k\} + \pi i \sum_{k\, along\, \mathcal{C}} Res\{f(z); z_k\}$$

- Gaussian integrals: $\int\limits_{-\infty}^{\infty} x^n e^{-ax^2/2}dx = \begin{cases} \frac{(n-1)!!}{a^{n/2}}\sqrt{\frac{2\pi}{a}} & \text{, n=even} \\ 0 & \text{, n=odd} \end{cases}$

  and $\int\limits_{-\infty}^{\infty} x^n e^{ibx^2/2}dx = \begin{cases} \frac{(n-1)!!}{(-ib)^{n/2}}\sqrt{\frac{2\pi}{|b|}} e^{i\frac{\pi}{4} sgn(b)} & \text{, n=even} \\ 0 & \text{, n=odd} \end{cases}$

- Stationary phase: x_o is a stationary point if $\phi'(x_o) = 0$

$$\int\limits_a^b A(x)e^{ik\phi(x)}dx = \sum_{j,\text{stat pts a,b}} A(x_j)e^{ik\phi(x_j)}\sqrt{\frac{2\pi}{k|\phi''(x_j)|}}e^{i\frac{\pi}{4}sgn(\phi''(x_j))} + \frac{A(x)}{ik\phi'(x)}e^{ik\phi(x)}\Big|_a^b$$

4 Integral transforms

- Functional analysis:
  - Cauchy-Schwartz inequality: $\|f\|^2\|g\|^2 \geq |\langle f, g\rangle|^2$ if $w(t) > 0$ and $\langle f, g\rangle = \int f^*(t)w(t)g(t)dt$. The equality is satisfied when f is proportional to g.
  - Kernel ($\sim$ matrix): $\bar{f}(\tau) = \int \mathcal{M}(\tau, t)f(t)dt$ equivalent to $A' = MA$ for matrices and vectors.
  - Unitary transformations: $\langle \bar{f}, \bar{g}\rangle = \langle f, g\rangle \Leftrightarrow \int \mathcal{U}^*(\tau, t_1)\mathcal{U}(\tau, t_2)d\tau = \delta(t_2 - t_1)$

- Fourier transform: $\mathcal{U}_F(p, x) = \sqrt{\frac{|\eta|}{2\pi}}e^{-i\eta xp}$

$$\tilde{f}(p) = \hat{\mathcal{F}}_{x\to p}f(x) = \sqrt{\frac{|\eta|}{2\pi}}\int\limits_{-\infty}^{\infty} f(x)e^{-i\eta xp}dx$$

$$f(x) = \hat{\mathcal{F}}^{-1}_{x\to p}\tilde{f}(p) = \sqrt{\frac{|\eta|}{2\pi}}\int\limits_{-\infty}^{\infty} \tilde{f}(p)e^{i\eta xp}dx$$

- Properties:
  - Shift and phase: $\hat{\mathcal{F}}_{x\to p}[f(x-a)] = \tilde{f}(p)e^{-i\eta ap}$ and $\hat{\mathcal{F}}_{x\to p}[f(x)e^{i\eta ap}] = \tilde{f}(p-a)$.
  - Scale: $\hat{\mathcal{F}}_{x\to p}[f(ax)] = \frac{1}{|a|}\tilde{f}(\frac{p}{a})$
  - Convolution and product: we define $f(x)*g(x) = \int f(x')g(x-x')dx'$ so we have that $\hat{\mathcal{F}}_{x\to p}[f(x)g(x)] = \sqrt{\frac{|\eta|}{2\pi}}(\tilde{f}*\tilde{g})(p)$ and $\hat{\mathcal{F}}_{x\to p}[(f*g)(x)] = \sqrt{\frac{2\pi}{|\eta|}}\tilde{f}(p)\tilde{g}(p)$
  - Derivative: $\hat{\mathcal{F}}_{x\to p}[f^{(n)}(x)] = (i\eta p)^n\tilde{f}(p)$
  - Parseval's theorem: $\int\limits_{-\infty}^{\infty} f^*(x)g(x)dx = \int\limits_{-\infty}^{\infty} \tilde{f}^*(p)\tilde{g}(p)dp$.
  - Uncertainty relation: we define the mean $\bar{x} = \frac{\int x|f(x)|^2dx}{\int |f(x)|^2dx}$ and the variance $\Delta x^2 = \frac{\int (x-\bar{x})^2|f(x)|^2dx}{\int |f(x)|^2dx}$, we have $\Delta x\Delta p \geq \frac{1}{2|\eta|}$.

- Laplace transform:

$$\bar{f}(s) = \hat{\mathcal{L}}_{t\to s} f(t) = \int_0^\infty f(t) e^{-st} dt \qquad t > 0,\ f \text{ can diverge as} t \to \infty$$

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \bar{f}(s) e^{st} ds \qquad \text{c large enough so that poles lie to the left of c}$$

- Properties with derivatives: $\hat{\mathcal{L}}_{t\to s} f^{(n)}(t) = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots f^{(n-1)}(0)$ used to solve ODE with constant coefficients.

- Hilbert transform: derived from $f(x) = -\frac{i}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{f(x')}{x'-x} dx'$ taking $f(x) = u(x) + iv(x)$

$$v(x) = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{u(x')}{x'-x} dx'$$

$$u(x) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{v(x')}{x'-x} dx'$$

- Titchmarsh theorem: the following statements are equivalent,

  1. $\tilde{f}(p) = 0\ \forall p < 0$ (assuming $\eta > 0$)
  2. $f(z)$ is analytic along and above the real axis and goes to zero as $|z| \to \infty$ within the upper half plane
  3. the real and imaginary parts of f are related via the Hilbert transform

5 Ordinary differential equations

- Second order: $y'' + py' + qy = -\rho$, we divide in the homogeneous and particular solution $y = \phi_h + \phi_p$, the homogeneous has two independent solutions ,$\phi_h = A_1\phi_1 + A_2\phi_2$, where the constants are chosen so that it satisfies the IC/BC and the particular is zero at the BC/IC.

- Wronskian: $W(x) = \phi_1\phi_2' - \phi_2\phi_1' = e^{-\int^x p(x')dx'}$

- Variational method: if ϕ_1 is known we suppose $\phi_2 = u\phi_1$ and so we get that $\phi_2 = \phi_1 \int^x \frac{W(x')}{\phi_1^2(x')} dx'$

- Frobenius method: works if p has up to a simple pole and q has up to a second order pole at x_o

  1. set $\phi(x) = \sum_{n=0}^{\infty} a_n (x - x_o)^{n+\gamma}$ where $a_0 = 1$
  2. expand p and q around x_o
  3. find equations in terms of a_n and γ
  4. determine $\gamma_{1,2}$ from the indicial equation (the first one, usually $n = 0$) and solve for a_n

  Note: second solution won't work if $\gamma_1 - \gamma_2 = 0$ or $N > 0$ integer, near x_o $\phi_1 \propto (x - x_o)^{\gamma_1}$ (same for ϕ_2 if the difference of the γ's is not an integer) and $\phi_2(x) \propto \begin{cases} (x - x_o)^{\gamma_2} & \text{if } \gamma_1 - \gamma_2 = N > 0 \\ (x - x_o)^{\gamma_1} \ln(x - x_o) & \text{if } \gamma_1 - \gamma_2 = 0 \end{cases}$

- Green function: $G(x,x') = \begin{cases} a\phi_1(x) + b\phi_2(x) & \text{if } x < x' \\ c\phi_1(x) + d\phi_2(x) & \text{if } x > x' \end{cases}$ G vanishes at BC/IC and satisfies the matching conditions $\begin{array}{rcl} a\phi_1(x') + b\phi_2(x') & = & c\phi_1(x') + d\phi_2(x') \\ a\phi_1'(x') + b\phi_2'(x') & = & c\phi_1'(x') + d\phi_2'(x') + 1 \end{array}$ for IC $a = b = 0$.

- Particular solution: $\phi_p(x) = \int_{\text{R.o.I}} \rho(x')G(x,x')dx'$

- Modes: solution to homogeneous equation that vanishes at the endpoints x_0 and x_1.

- Self-adjoint operators: $\langle v, \hat{\mathcal{L}}u \rangle = \langle u, \hat{\mathcal{L}}v \rangle^* + [\quad]_{x_0}^{x_1}$ if $\hat{\mathcal{L}} = p_0(x)\frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + p_2(x)$ then we must have $p_0' = p_1$ and we can write $\hat{\mathcal{L}}u = (\alpha u')' + \beta u$, if instead we have $\hat{\mathcal{D}}y = y'' + py' + qy$ which is not in self-adjoint form we let $y = \alpha u$ and so $\alpha = W$ and $\beta = (q - p')W$ ($\hat{\mathcal{D}}y = \hat{\mathcal{L}}u = 0$) and there can be no modes if $\alpha\beta \leq 0 \ \forall \ x \in [x_0, x_1]$.

6 Other useful stuff

- Geometric series: $\sum\limits_{n=0}^{\infty} w^n = \frac{1}{1-w}$ for $|w| < 1$

- Binomial expansion: $(1+w)^\alpha = \sum\limits_{n=0}^{\infty} \frac{\prod\limits_{p=0}^{n-1}(\alpha - p)}{n!} w^n$

- Partial fractions: when quadratic there are two terms.