1. Consider the Hamiltonian in one dimension

$$H = \frac{p^2}{2m} + V(x)$$

where

$$V(x) = -V_0\delta(x) + V'(x)$$

with

$$V'(x) = \begin{cases} V_1 & |x| < a \\ 0 & |x| > a \end{cases}$$

Take $V_0, V_1 > 0$.

a) How many bound states are possible? What are their parities?

b) Assuming such a bound state, write the form of the wave function in the various regions where $V(x)$ is constant.

c) Determine the complete wave function which satisfies the relevant boundary conditions for the case of a bound state of arbitrarily small energy. Sketch the wave function.

d) What is the relation between V_0 and V_1 which leads to such a (zero energy) state?

e) Does the Bargmann limit make any predictions for the given potential?

2. Consider three spin one-half systems described by angular momentum operators $\boldsymbol{S}_1$, $\boldsymbol{S}_2$ and $\boldsymbol{S}_3$. The Hamiltonian is

$$H = \lambda[(\boldsymbol{S}_1 \cdot \boldsymbol{S}_2)(\boldsymbol{S}_1 \cdot \boldsymbol{S}_3) + (\boldsymbol{S}_1 \cdot \boldsymbol{S}_3)(\boldsymbol{S}_1 \cdot \boldsymbol{S}_2)]$$

a) Give a CSCO which has H as one of its elements.

b) Show (or explain why) the elements of a) are mutually commutative.

c) Determine the eigenvalues of H and the multiplicity of each.

d) Using the notation $|+-+\rangle, |--+\rangle, \ldots$ construct the eigenkets and the CSCO eigenvalues corresponding to each.

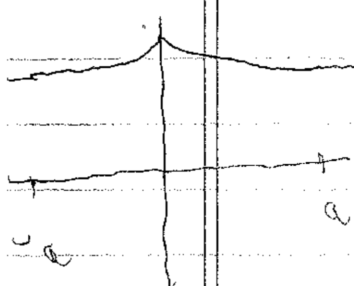
e) Consider the statement "The CSCO of a) can be replaced by a set in which H is replaced by a particular one of the exchange operators X_{ij}". True or False? Why? Note the definition

$$X_{ij}\boldsymbol{S}_i X_{ij}^{-1} = \boldsymbol{S}_j$$

1. a) Only one bound state possible because odd parity state does not "feel" the $\delta(x)$ term. Bound state is even parity if it exists. Note also that decreasing V_1 to zero (thereby making it more "attractive") reduces V to simply a delta function which can support only a single bound state.

b)
$$\psi = A e^{-\kappa|x|} \qquad |x| \geq a$$
$$E = -\frac{\hbar^2\kappa^2}{2m}$$
$$\psi(x) = B \sinh \kappa'|x| + C \cosh \kappa' x \qquad 0 \leq x \leq a$$
$$\frac{\hbar^2\kappa'^2}{2m} = \frac{\hbar^2\kappa^2}{2m} + V_1$$

c) for $\kappa \to 0$
$$\psi(x) \to A \qquad |x| \geq a$$
$$\psi(x) \to A \cosh \kappa'(|x| - a) \qquad 0 \leq x \leq a$$
$$\kappa' = \sqrt{\frac{2mV_1}{\hbar^2}}$$



d) BC
$$2A\kappa' \sinh \kappa' a = \frac{2mV_0}{\hbar^2} A \cosh \kappa' a$$
$$\kappa' a \tanh \kappa' a = \frac{mV_0}{\hbar^2}$$

e) Bargmann says nothing about even parity case. Concerning odd parity case note that
$$\int x\, V_{attractive} = 0$$
so no odd parity state.

2. a) H, S_1^2, S_2^2, S_3^2, $(S_1+S_2+S_3)^2$, $(S_1+S_2+S_3)_z$

b) As always for angular momenta

$$[(S_1+S_2+S_3)^2, \vec{S}_1+\vec{S}_2+\vec{S}_3] = 0$$

The S_1^2, S_2^2, S_3^2 are mutually commutative

Also $[S_1^2, (S_1+\cancel{S_2}+\cancel{S_3})_z] = [S_1^2, S_{1z}] = 0$

so $[S_1^2, (S_1+S_2+S_3)^2] = 0$ and also for S_2^2, S_3^2

Also $[S_1^2, H] = 0$ since S_1^2 commutes with every ~~term in H and also S_2^2, S_3^2.~~

Finally $[H, (S_1+S_2+S_3)_z] = 0$
~~Because H is rotationally invariant. Also~~

$$[H, (S_1+S_2+S_3)^2] = 0.$$

c) $H = \lambda(S_{2i} S_{3j} + S_{3i} S_{2j}) \underbrace{S_{1i} S_{1j}}_{\frac{\hbar^2}{4}\left(\delta_{ij} + i\epsilon_{ijk}\frac{2}{\hbar} S_{1k}\right)}$

$$= \lambda \frac{\hbar^2}{4}(2 S_2 \cdot S_3)$$

$$+ i\lambda \frac{\hbar^2}{4}(S_2 \times S_3 \cdot S_1 + S_3 \times S_2 \cdot S_1)\frac{2}{\hbar}$$

$$= \frac{\lambda\hbar^2}{2} S_2 \cdot S_3$$

$$= \frac{\lambda\hbar^2}{2}\,\frac{(S_2+S_3)^2 - S_2^2 - S_3^2}{2} = \frac{\lambda\hbar^2}{4}\left[(S_2+S_3)^2 - \frac{3}{2}\hbar^2\right]$$

$$= \frac{\lambda\hbar^4}{4}\left[(0, 2) - \frac{3}{2}\right]$$

$$= \frac{\lambda\hbar^4}{8}\begin{pmatrix} 1, & -3 \end{pmatrix}$$

two fold

four fold

d) $\frac{1}{\sqrt{2}}\left(|\pm + -\rangle - |\pm - +\rangle\right)$ two of these

$E = -\frac{3}{8}\lambda\hbar^4$, $S^2 = \frac{3}{4}\hbar^2$, $S_z = \pm\hbar/2$

also $|\pm\pm\pm\rangle$ ← 2 of these $E = \frac{1}{8}\lambda\hbar^4$, $S^2 = \frac{15}{4}\hbar^2$, $S_z = \pm\frac{3}{2}\hbar$

and

$\frac{1}{\sqrt{3}}\left(|\pm\pm\mp\rangle + |\pm\mp\pm\rangle + |\mp\pm\pm\rangle\right)$ ← 2 of these

$E = \frac{1}{8}\lambda\hbar^4$, $S^2 = \frac{15}{4}\hbar^2$, $S_z = \pm\frac{\hbar}{2}$

finally

$A\left(|\pm + -\rangle + |\pm - +\rangle\right) + B|\mp\pm\pm\rangle$

$2A + B = 0$ so

$A = \frac{1}{\sqrt{6}}$, $B = -\frac{2}{\sqrt{6}}$

$\frac{1}{\sqrt{6}}\left[|\pm + -\rangle + |\pm - +\rangle - 2|\mp\pm\pm\rangle\right]$

$E = \frac{1}{8}\lambda\hbar^4$, $S^2 = \frac{3}{4}\hbar^2$, $S_z = \pm\frac{1}{2}\hbar$

e) True because all the (six) $E = \frac{1}{8}\lambda\hbar^4$ eigenvalues correspond to eigenvalue $+1$ of X_{23} while the $E = -\frac{3}{8}\lambda\hbar^4$ correspond to eigenvalue -1 of X_{23}.