

**Physics 221B**  
**Spring 2011**  
**Notes 39**  
**The Quantized Electromagnetic Field**

In these notes we will quantize the electromagnetic field, working from the classical Hamiltonian description worked out in Notes 37. We will work primarily with the free field, and leave the interaction with matter for later sets of notes. These notes continue with the notation established in Notes 37.

We begin by applying Dirac's quantization prescription to the classical field-matter system discussed in Notes 37. At first we work with the free field only, whose classical Hamiltonian is

$$H_{\text{em}} = \frac{1}{2} \sum_{\lambda} \omega_k (P_{\lambda}^2 + Q_{\lambda}^2) \quad (1)$$

[see Eq. (37.75)]. We reinterpret the variables  $Q_{\lambda}$ ,  $P_{\lambda}$  as operators satisfying the commutation relations,

$$[Q_{\lambda}, P_{\lambda'}] = i\hbar \delta_{\lambda\lambda'}, \quad [Q_{\lambda}, Q_{\lambda'}] = [P_{\lambda}, P_{\lambda'}] = 0, \quad (2)$$

whereupon the Hamiltonian  $H_{\text{em}}$  also becomes an operator.

We see that the free field Hamiltonian is a sum of independent harmonic oscillators, one for each mode of the field, so we introduce the usual Dirac algebraic formalism for harmonic oscillators. First we define the annihilation and creation operators,

$$a_{\lambda} = \frac{Q_{\lambda} + iP_{\lambda}}{\sqrt{2\hbar}},$$

$$a_{\lambda}^{\dagger} = \frac{Q_{\lambda} - iP_{\lambda}}{\sqrt{2\hbar}}, \quad (3)$$

which are just like the classical formulas (37.100) except for the replacement of  $*$  by  $\dagger$ . These operators satisfy the commutation relations,

$$[a_{\lambda}, a_{\lambda'}^{\dagger}] = \delta_{\lambda\lambda'}, \quad [a_{\lambda}, a_{\lambda'}] = [a_{\lambda}^{\dagger}, a_{\lambda'}^{\dagger}] = 0, \quad (4)$$

as follows from Eq. (2). Next we write the free field Hamiltonian (1) in terms of the creation and annihilation operators,

$$H_{\text{em}} = \sum_{\lambda} \hbar\omega_k \left( a_{\lambda}^{\dagger} a_{\lambda} + \frac{1}{2} \right), \quad (5)$$

where the  $1/2$  represents the usual zero point energy of a harmonic oscillator. We also define the usual number operator,

$$N_{\lambda} = a_{\lambda}^{\dagger} a_{\lambda}. \quad (6)$$

What are the ket spaces upon which these operators act? It is possible to introduce a Hilbert space of wave functions  $\psi(Q_{\lambda})$  for each mode of the field, but in practice this is never done because

the algebraic relations among the operators and energy eigenkets are all that is ever needed. We will denote the energy eigenkets of a single oscillator  $\lambda$  by  $|n_\lambda\rangle$ , where  $n_\lambda = 0, 1, \dots$  is the usual quantum number of a harmonic oscillator. These kets span a space  $\mathcal{E}_\lambda$  associated with the single mode, which is the space upon which the operators  $Q_\lambda, P_\lambda$  (for the given value of  $\lambda$ ) act. The ket space for the entire electromagnetic field is the tensor product over the modes,

$$\mathcal{E}_{\text{em}} = \prod_{\lambda} \otimes \mathcal{E}_\lambda. \quad (7)$$

An arbitrary (pure) quantum state of the electromagnetic field is a ket in the space  $\mathcal{E}_{\text{em}}$ . The energy eigenstates of  $H_{\text{em}}$  are specified by a list of quantum numbers  $\{n_\lambda\} = \{\dots n_\lambda \dots\}$ , one for each mode of the field; the eigenstates themselves will be written in various ways,

$$|\{n_\lambda\}\rangle = |\dots n_\lambda \dots\rangle = \prod_{\lambda} \otimes |n_\lambda\rangle. \quad (8)$$

The creation/annihilation operators act on these eigenstates in the usual way,

$$\begin{aligned} a_\lambda |\dots n_\lambda \dots\rangle &= \sqrt{n_\lambda} |\dots n_\lambda - 1 \dots\rangle, \\ a_\lambda^\dagger |\dots n_\lambda \dots\rangle &= \sqrt{n_\lambda + 1} |\dots n_\lambda + 1 \dots\rangle. \end{aligned} \quad (9)$$

The ground state of the free field is the state with all  $n_\lambda = 0$ , which we denote simply by  $|0\rangle$ ,

$$|0\rangle = |\dots 0 \dots\rangle. \quad (10)$$

We call  $|0\rangle$  the *vacuum state*. It is not to be confused with the zero ket; the vacuum is a state of unit norm,  $\langle 0|0\rangle = 1$ . The vacuum ket is annihilated by any annihilation operator and the vacuum bra is annihilated by any creation operator,

$$a_\lambda |0\rangle = 0, \quad \langle 0|a_\lambda^\dagger = 0. \quad (11)$$

On the other hand, by applying creation operators to the vacuum, we can build up a state with any number of photons. This gives

$$|\dots n_\lambda \dots\rangle = \frac{1}{\sqrt{\prod_{\lambda} n_\lambda!}} \prod_{\lambda} (a_\lambda^\dagger)^{n_\lambda} |0\rangle, \quad (12)$$

which is a generalization of Eq. (8.38).

Since the vacuum is the state of minimum energy of the electromagnetic field, we interpret it physically as a state in which there are no photons. Unfortunately, according to Eq. (5), the energy of the vacuum is

$$\langle 0|H_{\text{em}}|0\rangle = \sum_{\lambda} \frac{\hbar\omega_k}{2} = \infty, \quad (13)$$

which is the infinite sum of the zero point energies of all the oscillators that make up the field. In the case of mechanical harmonic oscillators with a finite number of degrees of freedom, the zero point

energy is real and physically meaningful, but here in the case of the electromagnetic field it is an embarrassment that causes difficulties of physical interpretation.

The zero point energy is just one of several classes of infinities that arise in quantum field theory, and it is one of the easier ones to rationalize away. In one approach, we simply argue that the definition of the origin of energy is a matter of convention anyway, that it does not affect the equations of motion (either the classical Hamilton equations or the Heisenberg equations in quantum mechanics), and that we ought to be able to throw the zero point energy away since it is just a constant, albeit an infinite one. In another approach, we can argue that the zero point energy is connected with the ambiguities in the quantization of classical Hamiltonians containing nontrivial orderings of  $q$ 's and  $p$ 's. It is true that the Hamiltonian (1) does not have any  $q$ - $p$  products, but that is true only in the  $(Q_\lambda, P_\lambda)$  system of canonical coordinates. In another coordinate system there would be nontrivial products. For example, classically there is no difference between  $a_\lambda^* a_\lambda$  and  $a_\lambda a_\lambda^*$ , but of course in quantum mechanics  $a_\lambda^\dagger a_\lambda = a_\lambda a_\lambda^\dagger - 1$ . Therefore if we decided to quantize by replacing  $a$ 's and  $a^*$ 's by  $a$ 's and  $a^\dagger$ 's, the quantum Hamiltonian would depend on the ordering of the classical  $a$ 's and  $a^*$ 's. By using different orderings, we could get the Hamiltonian (5), or one having twice the zero point energy, or one with no zero point energy at all. Without further rationalization, we will choose the latter possibility, so that the Hamiltonian becomes

$$H_{\text{em}} = \sum_{\lambda} \hbar \omega_k a_\lambda^\dagger a_\lambda, \quad (14)$$

and so that the energy of the state  $|\dots n_\lambda \dots\rangle$  is given by

$$H_{\text{em}} |\dots n_\lambda \dots\rangle = \left( \sum_{\lambda} n_\lambda \hbar \omega_k \right) |\dots n_\lambda \dots\rangle, \quad (15)$$

or,

$$\langle \dots n_\lambda \dots | H_{\text{em}} | \dots n_\lambda \dots \rangle = \sum_{\lambda} n_\lambda \hbar \omega_k. \quad (16)$$

In particular, the energy of the vacuum is zero.

We saw classically that the normal variable  $a_\lambda$  is essentially the amplitude of a plane electromagnetic wave of mode  $\lambda = (\mathbf{k}, \mu)$ . Thus, the classical quantity  $|a_\lambda|^2 = a_\lambda^* a_\lambda$  is proportional to the energy in the mode. But in quantum mechanics, we are finding that the energy in mode  $\lambda$ , which is proportional to the expectation value of  $a_\lambda^\dagger a_\lambda$ , is quantized in units of  $\hbar \omega_k$ . This of course is exactly as in the original quantum hypothesis developed by Planck and Einstein, and we are led to interpret a state with quantum number  $n_\lambda$  as one containing  $n_\lambda$  photons, each with energy  $\hbar \omega_k$ .

This is the beginning of the particle interpretation of the quantum states that belong to the ket space  $\mathcal{E}_{\text{em}}$ . In the following discussion we will gradually flesh out this interpretation by examining successively the momentum, angular momentum, and spin and statistics of these particles. In the process, we will see that the formalism we have developed for the quantization of the field incorporates a quantum mechanical description of a system in which particles can be created or destroyed, so that the number of particles is variable. The particles in question are photons, which

can be created in arbitrary numbers at arbitrarily low energies, because they are massless. But the formalism we will develop serves as a paradigm for higher energy (relativistic) processes in which massive particles can be created or destroyed.

We turn now to the momentum of the photons. To investigate the momentum in the quantum theory, we must transcribe the classical momentum given by Eqs. (38.139) or (38.149) into a quantum operator. For the present we are working with the free field only, for which the matter terms can be dropped and for which  $\mathbf{E} = \mathbf{E}_\perp$ . Therefore the momentum of interest is  $\mathbf{P}_{\text{trans}}$ , which is expressed in terms of the fields  $\mathbf{E}_\perp$  and  $\mathbf{B}$  by

$$\mathbf{P}_{\text{trans}} = \frac{1}{4\pi c} \int d^3\mathbf{r} \mathbf{E}_\perp \times \mathbf{B} \quad (17)$$

[see Eq. (38.140)]. Classically, the fields are functions of the  $a$ 's and  $a^*$ 's; our strategy will be to transcribe these to  $a$ 's and  $a^\dagger$ 's to get a quantum operator.

We begin with the fields  $\mathbf{A}$ ,  $\mathbf{E}_\perp$  and  $\mathbf{B}$ , which are expressed classically in terms of  $a$ 's and  $a^*$ 's by Eqs. (38.128)–(38.130). When we transcribe these over to quantum mechanics, we obtain the operators,

$$\mathbf{A}(\mathbf{r}) = \sqrt{\frac{2\pi\hbar c^2}{V}} \sum_\lambda \frac{1}{\sqrt{\omega_k}} (\epsilon_\lambda a_\lambda e^{i\mathbf{k}\cdot\mathbf{r}} + \epsilon_\lambda^* a_\lambda^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}}), \quad (18)$$

$$\mathbf{E}_\perp(\mathbf{r}) = \frac{1}{c} \sqrt{\frac{2\pi\hbar c^2}{V}} \sum_\lambda \sqrt{\omega_k} (i\epsilon_\lambda a_\lambda e^{i\mathbf{k}\cdot\mathbf{r}} - i\epsilon_\lambda^* a_\lambda^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}}), \quad (19)$$

$$\mathbf{B}(\mathbf{r}) = \sqrt{\frac{2\pi\hbar c^2}{V}} \sum_\lambda \frac{1}{\sqrt{\omega_k}} [i(\mathbf{k} \times \epsilon_\lambda) a_\lambda e^{i\mathbf{k}\cdot\mathbf{r}} - i(\mathbf{k} \times \epsilon_\lambda^*) a_\lambda^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}}]. \quad (20)$$

Just as in classical field theory, the field point  $\mathbf{r}$  is regarded merely as a parameter of the fields, but now the fields themselves are operators, since the right hand sides are linear combinations of the operators  $a_\lambda$  and  $a_\lambda^\dagger$ . Thus,  $\mathbf{A}(\mathbf{r})$ ,  $\mathbf{E}_\perp(\mathbf{r})$ , and  $\mathbf{B}(\mathbf{r})$  are now seen as fields of operators that act on the ket space of the quantized system; they are our first examples of *quantum fields*. We note that just as the classical fields are real, these quantum fields are Hermitian. As usual in quantum mechanics, Hermitian operators correspond to measurements that can be made on a physical system, and such measurements are subject to statistical fluctuations and the constraints of the uncertainty principle. We will see that such features are present in the measurement of the quantized electromagnetic fields.

Let us return to the field momentum  $\mathbf{P}_{\text{trans}}$  in Eq. (17), which is the classical expression. The classical fields  $\mathbf{E}_\perp$  and  $\mathbf{B}$  that appear in the integrand can be expanded as linear combinations of  $a$ 's and  $a^*$ 's, according to Eqs. (38.129) and (38.130). But when we replace the  $a$ 's and  $a^*$ 's by  $a$ 's and  $a^\dagger$ 's in order to obtain the quantum expression for  $\mathbf{P}_{\text{trans}}$ , there arises the question of the proper ordering of the classical  $a$ 's and  $a^*$ 's. We could simply follow the ordering given by  $\mathbf{E}_\perp \times \mathbf{B}$ , but this as it turns out leads to an infinite zero point momentum, much like the zero point energy in Eq. (5).

We would like to have  $\langle 0 | \mathbf{P}_{\text{trans}} | 0 \rangle = 0$ , so the momentum of the vacuum would vanish. We can accomplish this if we order the  $a$ 's and  $a^*$ 's in the classical formula by placing all  $a^*$ 's to the left of all  $a$ 's, and then replacing the  $a$ 's and  $a^*$ 's by  $a$ 's and  $a^\dagger$ 's. Then when we compute the vacuum expectation value of  $\mathbf{P}_{\text{trans}}$ , there will always be an annihilation operator next to the vacuum ket  $|0\rangle$ , or a creation operator next to the vacuum bra  $\langle 0|$ , and the result will vanish.

The following notation is convenient for this purpose. If we have any polynomial in  $a$ 's and  $a^\dagger$ 's, we will define the *normal ordered* polynomial as the rearrangement obtained by moving all  $a^\dagger$ 's to the left and all  $a$ 's to the right. In this process, we discard any commutators of  $a$ 's and  $a^\dagger$ 's (effectively, we work with the classical expression, then replace  $a$ 's and  $a^*$ 's by  $a$ 's and  $a^\dagger$ 's.) If  $Q$  is such a polynomial, we denote its normal ordered rearrangement by  $:Q:$ . For example, we can write the quantum Hamiltonian of the free electromagnetic field as

$$H_{\text{em}} = \frac{1}{8\pi} \int d^3\mathbf{r} :E_\perp^2 + B^2:, \quad (21)$$

and the momentum of the free field is

$$\mathbf{P}_{\text{trans}} = \frac{1}{4\pi c} \int d^3\mathbf{r} :\mathbf{E}_\perp \times \mathbf{B}:. \quad (22)$$

Let us now express the free field momentum  $\mathbf{P}_{\text{trans}}$  in terms of  $a$ 's and  $a^\dagger$ 's. We substitute Eqs. (19) and (20) into Eq. (22), and obtain

$$\begin{aligned} \mathbf{P}_{\text{trans}} = & \frac{1}{4\pi c} \int d^3\mathbf{r} \frac{1}{c} \frac{2\pi\hbar c^2}{V} \sum_{\lambda\lambda'} \sqrt{\frac{\omega}{\omega'}} : [i\epsilon_{\lambda} a_\lambda e^{i\mathbf{k}\cdot\mathbf{r}} - i\epsilon_{\lambda}^* a_\lambda^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}}] \\ & \times [i(\mathbf{k}' \times \epsilon_{\lambda'}) a_{\lambda'} e^{i\mathbf{k}'\cdot\mathbf{r}} - i(\mathbf{k}' \times \epsilon_{\lambda'}^*) a_{\lambda'}^\dagger e^{-i\mathbf{k}'\cdot\mathbf{r}}] :, \end{aligned} \quad (23)$$

where  $\lambda = (\mathbf{k}\mu)$ ,  $\lambda' = (\mathbf{k}'\mu')$ , and  $\omega = \omega_k$ ,  $\omega' = \omega_{k'}$ . There are four major terms in this expression. Let us first consider the term involving the product  $a_\lambda a_{\lambda'}$ :

$$\begin{aligned} a_\lambda a_{\lambda'}\text{-term} = & -\frac{\hbar}{2V} \int d^3\mathbf{r} \sum_{\mathbf{k}\mathbf{k}'} \sum_{\mu\mu'} \sqrt{\frac{\omega}{\omega'}} [\epsilon_{\mathbf{k}\mu} \times (\mathbf{k}' \times \epsilon_{\mathbf{k}'\mu'})] a_{\mathbf{k}\mu} a_{\mathbf{k}'\mu'} e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}} \\ = & +\frac{\hbar}{2} \sum_{\mathbf{k}\mu\mu'} [\epsilon_{\mathbf{k}\mu} \times (\mathbf{k} \times \epsilon_{-\mathbf{k},\mu'})] a_{\mathbf{k}\mu} a_{-\mathbf{k},\mu'}, \end{aligned} \quad (24)$$

where we have used

$$\frac{1}{V} \int d^3\mathbf{r} e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}} = \delta_{\mathbf{k},-\mathbf{k}'}. \quad (25)$$

Note that after setting  $\mathbf{k}' = -\mathbf{k}$ , we have  $\omega' = \omega$ . Next we expand out the double cross product and use Eq. (38.75b), so that Eq. (24) becomes

$$a_\lambda a_{\lambda'}\text{-term} = \frac{\hbar}{2} \sum_{\mathbf{k}\mu\mu'} \mathbf{k} (\epsilon_{\mathbf{k}\mu} \cdot \epsilon_{-\mathbf{k},\mu'}) a_{\mathbf{k}\mu} a_{-\mathbf{k},\mu'} = -\frac{\hbar}{2} \sum_{\mathbf{k}\mu\mu'} \mathbf{k} (\epsilon_{-\mathbf{k},\mu'} \cdot \epsilon_{\mathbf{k}\mu}) a_{-\mathbf{k},\mu'} a_{\mathbf{k}\mu}, \quad (26)$$

where in the second equality we have replaced the dummy index of summation  $\mathbf{k}$  by  $-\mathbf{k}$ , and swapped the indices  $\mu$ ,  $\mu'$ . However, since  $[a_{\mathbf{k}\mu}, a_{-\mathbf{k},\mu}] = 0$ , the whole expression is equal to the negative of

itself, and therefore vanishes. In a similar manner we find that the term in Eq. (23) involving  $a_\lambda^\dagger a_{\lambda'}^\dagger$ , also vanishes.

This leaves the terms involving  $a_\lambda a_{\lambda'}^\dagger$  and  $a_\lambda^\dagger a_{\lambda'}$ , of which the first becomes  $a_\lambda^\dagger a_\lambda$  upon normal ordering. By an analysis similar to that above, this term is

$$\begin{aligned} a_{\lambda'}^\dagger a_\lambda\text{-term} &= \frac{\hbar}{2V} \int d^3\mathbf{r} \sum_{\mathbf{k}\mathbf{k}'} \sum_{\mu\mu'} \sqrt{\frac{\omega}{\omega'}} [\boldsymbol{\epsilon}_{\mathbf{k}\mu} \times (\mathbf{k}' \times \boldsymbol{\epsilon}_{\mathbf{k}'\mu'}^*)] a_{\mathbf{k}'\mu'}^\dagger a_{\mathbf{k}\mu} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} \\ &= \frac{\hbar}{2} \sum_{\mathbf{k}\mu\mu'} \boldsymbol{\epsilon}_{\mathbf{k}\mu} \times (\mathbf{k} \times \boldsymbol{\epsilon}_{\mathbf{k}\mu'}^*) a_{\mathbf{k}\mu'}^\dagger a_{\mathbf{k}\mu} \\ &= \frac{\hbar}{2} \sum_{\mathbf{k}\mu} \mathbf{k} a_{\mathbf{k}\mu}^\dagger a_{\mathbf{k}\mu} = \frac{1}{2} \sum_{\lambda} \hbar \mathbf{k} a_\lambda^\dagger a_\lambda, \end{aligned} \quad (27)$$

where we have used Eq. (38.75a) in expanding the double cross product. Similarly, the  $a_\lambda^\dagger a_{\lambda'}$  term gives the same answer, and doubles it. Altogether, we find

$$\mathbf{P}_{\text{trans}} = \sum_{\lambda} \hbar \mathbf{k} a_\lambda^\dagger a_\lambda = \sum_{\lambda} \hbar \mathbf{k} N_\lambda, \quad (28)$$

where  $N_\lambda$  is the number operator.

The momentum operator  $\mathbf{P}_{\text{trans}}$  commutes with the Hamiltonian  $H_{\text{em}}$ , and is diagonal in the basis  $\{|n_\lambda\rangle\}$  of energy eigenkets. Explicitly, we have

$$\mathbf{P}_{\text{trans}} |\dots n_\lambda \dots\rangle = \left( \sum_{\lambda} n_\lambda \hbar \mathbf{k} \right) |\dots n_\lambda \dots\rangle. \quad (29)$$

We see that, just as the energy of a mode of the field is quantized in integer multiples of  $\hbar\omega_k$ , the momentum in the mode is quantized in integer multiples of  $\hbar\mathbf{k}$ . We interpret this by saying that the photon is a particle of energy  $\hbar\omega_k$  and momentum  $\hbar\mathbf{k}$ , which is completely in accordance with the dispersion relation  $\omega = ck$  for a light wave, as well as the relativistic energy-momentum relation  $E = cp$  for a massless particle.

We turn now to the angular momentum of the photon. This is a somewhat complicated subject, and we can give only a partial analysis here. The principal conclusions from a complete analysis of this subject are that the photon is a particle of spin 1, but that of the three helicity states ( $\mu = 0, \pm 1$ ) that would be present for a massive particle, the photon has only two ( $\mu = \pm 1$ ). The latter restriction is equivalent to the transversality condition imposed on the radiation fields, and it is a general feature of massless particles: as first shown by Wigner, massless particles of spin  $s$  only possess the stretched helicity states  $\mu = \pm s$ .

For the analysis of the angular momentum of the field, a particular choice of polarization vectors is convenient. These are essentially circular polarization vectors, with a particular phase convention. In general, polarization vectors are two orthonormal, possibly complex unit vectors that span the plane perpendicular to  $\mathbf{k}$ . These vectors are, of course, functions of  $\mathbf{k}$ , or, more precisely, of the direction  $\hat{\mathbf{k}}$ . A convenient way to construct such vectors is to start with two constant, orthonormal

unit vectors that span the  $x$ - $y$  plane, so that taken with  $\hat{\mathbf{z}}$  they form an orthonormal triad. Then the vectors of this triad are rotated by a rotation matrix  $\mathbf{R}$  that is required to map the  $\hat{\mathbf{z}}$  direction into the  $\hat{\mathbf{k}}$  direction. We write  $\mathbf{R}(\hat{\mathbf{k}})$  for this matrix; it is a function of  $\hat{\mathbf{k}}$ , and by its definition we have

$$\mathbf{R}(\hat{\mathbf{k}})\hat{\mathbf{z}} = \hat{\mathbf{k}}. \quad (30)$$

Such a matrix is easy to construct; if the spherical angles of  $\hat{\mathbf{k}}$  are  $(\theta, \phi)$ , then we will take

$$\mathbf{R}(\hat{\mathbf{k}}) = \mathbf{R}_z(\phi)\mathbf{R}_y(\theta) = \mathbf{R}(\phi, \theta, 0), \quad (31)$$

where the final expression is in terms of Euler angles. When the two vectors of the triad that span the  $x$ - $y$  plane are rotated by  $\mathbf{R}(\hat{\mathbf{k}})$ , they become two vectors that span the plane perpendicular to  $\hat{\mathbf{k}}$ , since the orthonormality conditions are preserved by the rotation.

In particular, suppose we take the original triad to be the spherical basis of unit vectors, introduced in Eq. (18.28), which we reproduce here:

$$\begin{aligned} \hat{\mathbf{e}}_1 &= -\frac{\hat{\mathbf{x}} + i\hat{\mathbf{y}}}{\sqrt{2}}, \\ \hat{\mathbf{e}}_0 &= \hat{\mathbf{z}}, \\ \hat{\mathbf{e}}_{-1} &= \frac{\hat{\mathbf{x}} - i\hat{\mathbf{y}}}{\sqrt{2}}. \end{aligned} \quad (32)$$

For light propagating in the  $z$ -direction, the vector  $\hat{\mathbf{e}}_1$  corresponds to left circular polarization (the electric field vector rotates counterclockwise in the  $x$ - $y$  plane), and the vector  $\hat{\mathbf{e}}_{-1}$  corresponds to right circular polarization light (the electric vector rotates clockwise). These are the conventions used by Jackson and Born and Wolf and most people in optics, but they are the opposite to what particle physicists would have preferred if they could have established the convention. Apparently for this reason, Sakurai has reversed the conventions. I think it is less confusing to stay with the standard terminology of optics. We can also invent new terminology; as we will see, left circular polarization can also be called the  $+1$  state of helicity, and right circular polarization the state of  $-1$  helicity. Helicity will be explained more fully below.

Given the spherical basis (32), we can define a rotated triad by

$$\boldsymbol{\epsilon}_{\mathbf{k}\mu} = \mathbf{R}(\hat{\mathbf{k}})\hat{\mathbf{e}}_\mu, \quad (33)$$

so that  $\boldsymbol{\epsilon}_{\mathbf{k}0} = \hat{\mathbf{k}}$ , and  $\boldsymbol{\epsilon}_{\mathbf{k}\mu}$  for  $\mu = \pm 1$  span the plane perpendicular to  $\hat{\mathbf{k}}$  and represent states of circular polarization for waves propagating in the  $\hat{\mathbf{k}}$  direction. We will henceforth take the index  $\mu$  to run over  $\pm 1$  for these polarization vectors (not 1 and 2, as in Notes 38). In addition to the orthonormality and completeness relations (38.75), one can show that these vectors also satisfy the relation  $\boldsymbol{\epsilon}_{\mathbf{k}\mu} = \boldsymbol{\epsilon}_{-\mathbf{k},\mu}^*$  and the relations

$$\boldsymbol{\epsilon}_{\mathbf{k}\mu} = (-1)^\mu \boldsymbol{\epsilon}_{\mathbf{k},-\mu}^*, \quad (34)$$

and

$$\boldsymbol{\epsilon}_{\mathbf{k}\mu}^* \times \boldsymbol{\epsilon}_{\mathbf{k}\mu} = i\mu\hat{\mathbf{k}}, \quad (35)$$

which are valid for  $\mu = 0, \pm 1$ . These relations are easily proved because they are just rotated versions of the analogous relations for the constant vectors  $\hat{\mathbf{e}}_\mu$ .

Let us return to the angular momentum of the system. The classical angular momentum of the matter-field system is given by Eqs. (38.155)–(38.161). We specialize this to the case of the free field, for which

$$\mathbf{J} = \mathbf{J}_{\text{trans}} = \mathbf{L} + \mathbf{S} = \frac{1}{4\pi c} \int d^3\mathbf{r} \mathbf{r} \times (\mathbf{E}_\perp \times \mathbf{B}). \quad (36)$$

We will transcribe the expressions for  $\mathbf{L}$  and  $\mathbf{S}$  over into quantum operators. We begin with the spin [see Eq. (38.157)], which we write as a normal ordered operator,

$$\begin{aligned} \mathbf{S} &= -\frac{1}{4\pi c} \int d^3\mathbf{r} : \mathbf{A} \times \mathbf{E}_\perp : \\ &= -\frac{\hbar}{2V} \int d^3\mathbf{r} \sum_{\lambda\lambda'} \sqrt{\frac{\omega'}{\omega}} : [\epsilon_\lambda a_\lambda e^{i\mathbf{k}\cdot\mathbf{r}} + \epsilon_\lambda^* a_\lambda^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}}] \\ &\quad \times [i\epsilon_{\lambda'} a_{\lambda'} e^{i\mathbf{k}'\cdot\mathbf{r}} - i\epsilon_{\lambda'}^* a_{\lambda'}^\dagger e^{-i\mathbf{k}'\cdot\mathbf{r}}] :, \end{aligned} \quad (37)$$

and then we evaluate terms as we did previously for the momentum  $\mathbf{P}_{\text{trans}}$ . As before, we find that the terms involving  $a_\lambda a_{\lambda'}$  and  $a_\lambda^\dagger a_{\lambda'}^\dagger$  vanish, while the remaining two cross terms give equal contributions. The answer can be written in the form,

$$\mathbf{S} = i\hbar \sum_{\mathbf{k}\mu\mu'} \epsilon_{\mathbf{k}\mu} \times \epsilon_{\mathbf{k}\mu'}^* a_{\mathbf{k}\mu'}^\dagger a_{\mathbf{k}\mu}, \quad (38)$$

which is valid for any choice of polarization vectors. Of course, the  $\mu$  sum only runs over the transverse polarizations. If, however, we make the choice (33), then the cross product vanishes unless  $\mu = \mu'$ , and we can use the identity (35) to write

$$\mathbf{S} = \sum_{\mathbf{k}\mu} \hbar \hat{\mathbf{k}}_\mu a_{\mathbf{k}\mu}^\dagger a_{\mathbf{k}\mu} = \sum_{\mathbf{k}} \hbar \hat{\mathbf{k}} (a_{\mathbf{k}+}^\dagger a_{\mathbf{k}+} - a_{\mathbf{k}-}^\dagger a_{\mathbf{k}-}) = \sum_{\mathbf{k}} \hbar \hat{\mathbf{k}} (N_{\mathbf{k}+} - N_{\mathbf{k}-}), \quad (39)$$

where  $N_{\mathbf{k}\pm}$  are the number operators for  $\mu = \pm 1$ .

Just like the energy  $H_{\text{em}}$  and momentum  $\mathbf{P}_{\text{trans}}$ , the spin  $\mathbf{S}$  can be expressed purely in terms of number operators, so it commutes with the free-field Hamiltonian and is diagonal in the occupation number basis  $|\dots n_\lambda \dots\rangle$ . We see that photons of  $\mu = \pm 1$  contribute to the angular momentum of the system an amount that is  $\hbar$  times  $\pm 1$  in the direction of the propagation. As we say, such photons have *helicity* of  $\pm 1$ .

Next we turn to the orbital angular momentum of the field. Here we cannot use box normalization any longer, because boxes are not invariant under rotations. To see the complete invariance of the electromagnetic matter-field system under rotations, it is necessary to take the limit  $V \rightarrow \infty$ , which of course means that the lattice in  $\mathbf{k}$ -space is replaced by a continuum of  $\mathbf{k}$ -values. We recall the rules (38.69)–(38.71) given earlier for the limit  $V \rightarrow \infty$ , and we begin by applying them to the Fourier expansions (18)–(20) for the fields  $\mathbf{A}$ ,  $\mathbf{E}_\perp$ , and  $\mathbf{B}$ . First we change notation,

$$\epsilon_{\mathbf{k}\mu} \rightarrow \epsilon_\mu(\mathbf{k}), \quad (40)$$



which is merely a way of reminding ourselves that  $\mathbf{k}$  is now a continuous variable. Next we note that the annihilation operator  $a_{\mathbf{k}\mu}$  is like a Fourier coefficient in  $\mathbf{k}$ -space of the field  $\mathbf{A}(\mathbf{r})$ , so we use the rule (38.70) for it, and write

$$a_{\mathbf{k}\mu} \rightarrow \frac{(2\pi)^{3/2}}{\sqrt{V}} a_{\mu}(\mathbf{k}). \quad (41)$$

With these changes, the quantum fields become

$$\mathbf{A}(\mathbf{r}) = \sqrt{2\pi\hbar c^2} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{\omega_k}} \sum_{\mu} [\epsilon_{\mu}(\mathbf{k}) a_{\mu}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} + \epsilon_{\mu}^*(\mathbf{k}) a_{\mu}^{\dagger}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}}], \quad (42)$$

$$\begin{aligned} \mathbf{E}_{\perp}(\mathbf{r}) &= \frac{1}{c} \sqrt{2\pi\hbar c^2} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \sqrt{\omega_k} \\ &\quad \times \sum_{\mu} [i\epsilon_{\mu}(\mathbf{k}) a_{\mu}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} - i\epsilon_{\mu}^*(\mathbf{k}) a_{\mu}^{\dagger}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}}], \end{aligned} \quad (43)$$

$$\begin{aligned} \mathbf{B}(\mathbf{r}) &= \sqrt{2\pi\hbar c^2} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{\omega_k}} \\ &\quad \times \sum_{\mu} \{i[\mathbf{k} \times \epsilon_{\mu}(\mathbf{k})] a_{\mu}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} - i[\mathbf{k} \times \epsilon_{\mu}^*(\mathbf{k})] a_{\mu}^{\dagger}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}}\}. \end{aligned} \quad (44)$$

The commutation relations (4) also change; now we have

$$[a_{\mu}(\mathbf{k}), a_{\mu'}^{\dagger}(\mathbf{k}')] = \delta_{\mu\mu'} \delta(\mathbf{k} - \mathbf{k}'), \quad [a_{\mu}(\mathbf{k}), a_{\mu'}(\mathbf{k}')] = [a_{\mu}^{\dagger}(\mathbf{k}), a_{\mu'}^{\dagger}(\mathbf{k}')] = 0. \quad (45)$$

When we go over to the continuum limit ( $V \rightarrow \infty$ ), the operators  $a_{\mu}(\mathbf{k})$  and  $a_{\mu}^{\dagger}(\mathbf{k})$  become singular, and have physical meaning only when used in appropriate expressions that are integrated over  $\mathbf{k}$ -space. It is easy to see why. When we were working in a box, a single mode was represented by a given plane light wave that was periodic in the box. When we quantize this mode and place, say, one photon in it, we have energy  $\hbar\omega$  in volume  $V$ , so the amplitude of the wave (speaking in classical terms) is finite, and the energy density is nonzero. When we go over to the continuum limit, however, the volume becomes infinite so the energy density corresponding to any finite number of photons in a single mode is zero. The energy of a single photon is still  $\hbar\omega$ , but if it is placed into a single mode, the energy is spread over all of space. Therefore if we want to obtain a localized distribution of energy, we must form linear combinations of different photon states with different  $\mathbf{k}$  values. This will in practice always turn into some kind of integral over  $\mathbf{k}$ -space. It is in this sense that the Dirac delta function occurring in the commutator (45) should be interpreted; of course, delta functions only have meaning when used under integral signs.

There is one more change of notation that is useful when discussing the angular momentum of the field. We define vector fields of annihilation/creation operators,

$$\begin{aligned} \mathbf{a}(\mathbf{k}) &= \sum_{\mu} \epsilon_{\mu}(\mathbf{k}) a_{\mu}(\mathbf{k}), \\ \mathbf{a}^{\dagger}(\mathbf{k}) &= \sum_{\mu} \epsilon_{\mu}^*(\mathbf{k}) a_{\mu}^{\dagger}(\mathbf{k}), \end{aligned} \quad (46)$$

which satisfy the commutation relations,

$$\begin{aligned} [a_i(\mathbf{k}), a_j^\dagger(\mathbf{k}')] &= T_{ij}(\mathbf{k}) \delta(\mathbf{k} - \mathbf{k}'), \\ [a_i(\mathbf{k}), a_j(\mathbf{k}')] &= [a_i^\dagger(\mathbf{k}), a_j^\dagger(\mathbf{k}')] = 0, \end{aligned} \quad (47)$$

where  $i, j$  refer to the Cartesian components and where  $T_{ij}(\mathbf{k})$  is the transverse projection tensor in  $\mathbf{k}$ -space,

$$T_{ij}(\mathbf{k}) = \delta_{ij} - \frac{k_i k_j}{k^2}. \quad (48)$$

Fields  $\mathbf{a}(\mathbf{k})$  and  $\mathbf{a}^\dagger(\mathbf{k})$  are transverse quantum fields in  $\mathbf{k}$ -space.

For reference, we now write out the energy, momentum and spin of the field in the new (continuum) language:

$$H_{\text{em}} = \int d^3\mathbf{k} \hbar\omega_k \sum_{\mu} a_{\mu}^\dagger(\mathbf{k}) a_{\mu}(\mathbf{k}) = \int d^3\mathbf{k} \hbar\omega_k \mathbf{a}^\dagger(\mathbf{k}) \cdot \mathbf{a}(\mathbf{k}), \quad (49)$$

$$\mathbf{P}_{\text{trans}} = \int d^3\mathbf{k} \hbar\mathbf{k} \sum_{\mu} a_{\mu}^\dagger(\mathbf{k}) a_{\mu}(\mathbf{k}) = \int d^3\mathbf{k} \hbar\mathbf{k} \mathbf{a}^\dagger(\mathbf{k}) \cdot \mathbf{a}(\mathbf{k}), \quad (50)$$

$$\mathbf{S} = \int d^3\mathbf{k} \hbar\hat{\mathbf{k}} \sum_{\mu} \mu a_{\mu}^\dagger(\mathbf{k}) a_{\mu}(\mathbf{k}) = -i\hbar \int d^3\mathbf{k} \mathbf{a}^\dagger(\mathbf{k}) \times \mathbf{a}(\mathbf{k}). \quad (51)$$

In the first expression for  $\mathbf{S}$ , we must use the circular or helicity basis of polarization vectors (33), but any polarization vectors can be used in the other expressions.

We return now to the orbital angular momentum of the field. We take the classical expression (38.156) and transcribe it into a normal ordered operator,

$$\mathbf{L} = \frac{1}{4\pi c} \int d^3\mathbf{r} \mathbf{r} \times (\nabla \mathbf{A} \cdot \mathbf{E}_{\perp}), \quad (52)$$

and then we express the integrand in terms of creation and annihilation operators. After simplification, we find an expression for the  $i$ -th component of the orbital angular momentum operator of the free field,

$$L_i = \frac{i\hbar}{2} \epsilon_{ij\ell} \int d^3\mathbf{k} k_{\ell} \left[ \mathbf{a}^\dagger(\mathbf{k}) \cdot \frac{\partial \mathbf{a}(\mathbf{k})}{\partial k_j} - \frac{\partial \mathbf{a}^\dagger(\mathbf{k})}{\partial k_j} \cdot \mathbf{a}(\mathbf{k}) \right]. \quad (53)$$

We see that the orbital angular momentum of the field is not expressed in terms of number operators, nor is it diagonal in the occupation number  $|\dots n_{\lambda} \dots\rangle$  basis. This should not be surprising; the modes we have been dealing with are plane waves at the classical level, and we know that plane waves in the quantum mechanics of a single particle are not generally angular momentum eigenstates. Instead, the nonrelativistic free particle eigenfunctions that are also eigenfunctions of  $L^2$  and  $L_z$  are spherical Bessel functions times  $Y_{\ell m}$ 's, times a spinor if the particle has spin. Something like this (but more complicated) is going on here with photon states; it is possible to organize photon states as eigenstates of the angular momentum operators, but our plane wave formalism developed so far has not done this. The subject of the angular momentum of the photon is somewhat lengthy, so we will not go into it further at this point, but some additional remarks will be made below.

Instead, we will discuss a striking aspect of the quantized field formalism we are developing, namely the fact that it describes the quantum mechanics of a system in which the number of particles is variable. We are accustomed to describing the state of a single particle system of spin  $s$  by a wave function  $\psi(\mathbf{r}, m)$ ; this is in the  $(\mathbf{r}, S_z)$ -representation, so  $m = -s, \dots, +s$ . Similarly, we are accustomed to describing the state of a two-particle system by a wave function  $\psi(\mathbf{r}_1, m_1; \mathbf{r}_2, m_2)$ , which if the particles are identical must obey the symmetry requirement,

$$\psi(\mathbf{r}_1, m_1; \mathbf{r}_2, m_2) = \pm \psi(\mathbf{r}_2, m_2; \mathbf{r}_1, m_1) \quad (54)$$

(+ for bosons,  $-$  for fermions). But in the quantized electromagnetic field, we describe photon states by the kets  $|\dots n_\lambda \dots\rangle$ , in which the number of photons in each mode is indicated by the integers  $n_\lambda$ . These occupation numbers can take on any value  $n_\lambda = 0, 1, 2, \dots$ , so the ket space  $\mathcal{E}_{\text{em}}$ , which is spanned by the occupation number basis kets  $|\dots n_\lambda \dots\rangle$ , includes states for any number of photons. It also includes states that are linear combinations of states of different numbers of photons.

What is the relation between the occupation number basis states  $|\dots n_\lambda \dots\rangle$  and the usual wave functions we are familiar with? Let us first consider the kets in  $\mathcal{E}_{\text{em}}$  that contain no photons. The only state with no photons is the vacuum  $|0\rangle$ , which spans a 1-dimensional subspace of  $\mathcal{E}_{\text{em}}$ .

Next we consider states of a single photon. A particular single photon state can be created by applying a creation operator to the vacuum, which gives  $a_\mu^\dagger(\mathbf{k})|0\rangle$  for some  $\mu$  and  $\mathbf{k}$ . But this is not the most general single photon state, which would be a linear combination of states of the form  $a_\mu^\dagger(\mathbf{k})|0\rangle$ , with different values of  $\mu$  and  $\mathbf{k}$ . We write such a state in the form,

$$|\Psi_1\rangle = \int d^3\mathbf{k} \sum_{\mu} f(\mathbf{k}, \mu) a_\mu^\dagger(\mathbf{k})|0\rangle, \quad (55)$$

where the 1-subscript on  $\Psi$  simply means that we have a single-photon state, and where the function  $f(\mathbf{k}, \mu)$  specifies the linear combination. The function  $f(\mathbf{k}, \mu)$  is an arbitrary complex function, apart from the condition

$$\int d^3\mathbf{k} \sum_{\mu=\pm 1} |f(\mathbf{k}, \mu)|^2 = 1, \quad (56)$$

which is required to make  $\langle\Psi_1|\Psi_1\rangle = 1$ . Conversely, given a single photon state  $|\Psi_1\rangle$ , we can solve for  $f$  by using

$$f(\mathbf{k}, \mu) = \langle 0|a_\mu(\mathbf{k})|\Psi_1\rangle, \quad (57)$$

as follows from the commutation relations (45). We see that (normalized) single particle photon states in  $\mathcal{E}_{\text{em}}$  can be placed into one-to-one correspondence with (normalized) wave functions  $f(\mathbf{k}, \mu)$ , where  $\mu = \pm 1$ . We will call  $f(\mathbf{k}, \mu)$  the “wave function of the photon.”

Similarly, we can create two-photon states by applying two creation operators to the vacuum, say,  $a_\mu^\dagger(\mathbf{k})a_{\mu'}^\dagger(\mathbf{k}')|0\rangle$ . An arbitrary two-photon state is a linear combination of such states,

$$|\Psi_2\rangle = \int d^3\mathbf{k} d^3\mathbf{k}' \sum_{\mu\mu'} f(\mathbf{k}, \mu; \mathbf{k}', \mu') a_\mu^\dagger(\mathbf{k})a_{\mu'}^\dagger(\mathbf{k}')|0\rangle. \quad (58)$$

We will interpret  $f(\mathbf{k}, \mu; \mathbf{k}', \mu')$  as the wave function of the two-photon state. However, since the two creation operators in Eq. (58) commute with one another, they could be applied in the opposite order, with no change to the state  $|\Psi_2\rangle$ . Therefore the function  $f$  might as well be symmetric in the arguments  $(\mathbf{k}, \mu)$  and  $(\mathbf{k}', \mu')$ ,

$$f(\mathbf{k}, \mu; \mathbf{k}', \mu') = f(\mathbf{k}', \mu'; \mathbf{k}, \mu), \quad (59)$$

because any antisymmetric part would not contribute to  $|\Psi_2\rangle$ . As a result, we reach the important conclusion that *photons are bosons*.

When we work with wave functions for identical particles, it is possible to write down a wave function that is not properly symmetrized (or antisymmetrized), even though such functions have no physical meaning. But when we specify two-photon states by means of creation operators applied to the vacuum, the states are always properly symmetrized, since the symmetrization is automatically built into the commutation relations of the creation operators. The same holds for states of any number of photons ( $n > 2$ ).

There is a similar formalism that works for fermions, and which automatically gives properly antisymmetrized states. This formalism also uses creation and annihilation operators, but the operators are required to satisfy anticommutation relations, instead of commutation relations. We will consider fermion fields later in the course.

Let us denote the subspace of  $\mathcal{E}_{\text{em}}$  spanned by the vacuum state by  $\mathcal{E}_0$ , the subspace spanned by all one-photon states by  $\mathcal{E}_1$ , the subspace spanned by all two-photon states by  $\mathcal{E}_2$ , etc. Then we have

$$\mathcal{E}_{\text{em}} = \mathcal{E}_0 \oplus \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \dots \quad (60)$$

Operators that act on  $\mathcal{E}_{\text{em}}$  that have matrix elements connecting the different subspaces  $\mathcal{E}_n$  are capable of changing the number of photons; such operators include the creation and annihilation operators  $a_\mu(\mathbf{k})$  and  $a_\mu^\dagger(\mathbf{k})$ , as well as the field operators  $\mathbf{A}(\mathbf{r})$ , etc. The free field Hamiltonian  $H_{\text{em}}$  does not change the number of photons, but when the matter Hamiltonian is included, the number of photons can change. Thus, in interactions with matter, the number of photons can increase or decrease; this is otherwise just the process of emission and absorption of radiation by matter, which we will consider in detail later.

The ket space  $\mathcal{E}_{\text{em}}$  is an example of a *Fock space*. There is a technical mathematical distinction between a Fock space and a Hilbert space that need not concern us;† we will simply use these terms for linguistic relief, with a Fock space designating a ket space incorporating a variable number of particles, such as  $\mathcal{E}_{\text{em}}$ , and with a Hilbert space designating the ket space with a fixed number of particles. For example, the wave functions  $f(\mathbf{k}, \mu)$  introduced in Eq. (55) belong to a Hilbert space of wave functions, while the ket  $|\Psi_1\rangle$  in that equation belongs to the Fock space  $\mathcal{E}_{\text{em}}$ .

In the case of a massive, nonrelativistic particle, the most popular form of the wave function is probably  $\psi(\mathbf{r}, m) = \langle \mathbf{r}, m | \psi \rangle$ , which is in the  $(\mathbf{r}, S_z)$ -representation. Of course, we are free to use

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† A Hilbert space has a countable basis; a Fock space does not.

other representations if we want to. Now we are calling  $f(\mathbf{k}, \mu)$  the wave function of the photon, but what is the representation? It turns out that it is the  $(\mathbf{k}, \Omega)$ -representation, where

$$\Omega = \hat{\mathbf{k}} \cdot \mathbf{S} \quad (61)$$

is the *helicity* operator. Also, it turns out that the wave function  $f(\mathbf{k}, \mu)$  represents the state of a spin-1 particle.

In the following discussion it will be important to distinguish between the Fock space  $\mathcal{E}_{\text{em}}$ , the ket space for the electromagnetic field, and the Hilbert space of wave functions of a single particle. Let us begin with the Hilbert space of a massive particle of spin  $s$ , which can be identified with the space of wave functions  $\psi(\mathbf{r}, m)$ . The vector  $\mathbf{r}$  is the usual position operator that acts on this space, and the vector  $\mathbf{k} = \mathbf{p}/\hbar$  is proportional to the usual momentum operator. For example, in the  $(\mathbf{r}, S_z)$ -representation,  $\mathbf{r}$  is represented by multiplication by  $\mathbf{r}$ , and  $\mathbf{k}$  is represented by  $-i\nabla$ . The operators  $\mathbf{r}$  and  $\mathbf{k}$ , which act on the Hilbert space of a single particle, are not to be confused with the  $\mathbf{r}$  and  $\mathbf{k}$  which occur in the theory of the quantized field, which are merely labels of the degrees of freedom of the field, and are not operators. In fact, there is no position operator for the field, and although the field does have a momentum operator [see Eq. (50)], it is quite different from the operator  $\mathbf{k}$  that acts on the single-particle Hilbert space. Similarly, we have the usual orbital angular momentum  $\mathbf{L} = \mathbf{r} \times \mathbf{p}$  and the usual spin  $\mathbf{S}$  that act on the Hilbert space of a single particle, but these are not to be confused with the orbital and spin angular momentum operators for the field [see Eqs. (51) and (53)], which act on  $\mathcal{E}_{\text{em}}$ . Finally, the helicity operator  $\Omega = \hat{\mathbf{k}} \cdot \mathbf{S}$  acts on the Hilbert space of a single particle; it is not a Fock space operator.

The helicity operator is just the component of the spin in a certain direction (the direction of propagation), so its eigenvalue  $\mu$  is like a magnetic quantum number, and takes on the values  $\mu = -s, \dots, +s$ . At least, this is the case for a particle of nonzero mass, the only case we have considered so far in this course. But it was shown by Wigner in 1939 that massless particles only have the stretched helicity states,  $\mu = \pm s$ . For example, the photon, with  $s = 1$ , only possesses the  $\mu = \pm 1$  states, and the graviton, another massless particle with  $s = 2$ , only possesses the  $\mu = \pm 2$  states. As we have seen, the exclusion of the  $\mu = 0$  states for the photon is equivalent to the transversality condition for the fields. But if photons had a nonzero mass, then they would also possess longitudinal polarizations, and all three helicity states would be allowed. Wigner's result can be understood more fully in terms of Lorentz transformations; if a particle has a nonzero mass, then it is always possible to go to the rest frame of the particle, whereupon ordinary spatial rotations can rotate the spin into any direction. But a massless particle has no rest frame.

In the case of a photon, the helicity states  $\mu = 0$  which would be allowed for a massive particle are simply not present. This means that the physical Hilbert space of wave functions for a photon is only a subspace of the space that would be allowed for a massive particle, and that any (Hilbert space) operator that has nonvanishing matrix elements between the  $\mu = \pm 1$  subspaces and the  $\mu = 0$  subspace must be regarded as nonphysical for a photon, since it would map a physically meaningful photon state into a physically meaningless state. As a result, we can classify the operators that act

on the state of a massive particle into those that are or are not meaningful when the mass is set to zero. Certainly any operator that commutes with helicity will not mix the eigenspaces of helicity, and so is meaningful when acting on photon states. This includes helicity  $\Omega$  itself, as well as the momentum  $\mathbf{k}$ . However, the position operator  $\mathbf{r}$  does not commute with helicity (because it does not commute with  $\mathbf{k}$ ), and is not meaningful for a photon. Thus, the photon does not have a position operator. An eigenfunction of position is a  $\delta$ -function, but such a function is not transverse. If we project out the transverse part, we get the transverse delta-function (38.48), which is not localized.

Nor does the photon have an orbital angular momentum operator, because the (Hilbert space) angular momentum  $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ , which is defined for a massive particle, does not commute with helicity  $\hat{\mathbf{k}} \cdot \mathbf{S}$  ( $\mathbf{L}$  generates spatial rotations, which rotate the  $\hat{\mathbf{k}}$  part of the dot product, but leave the  $\mathbf{S}$  part alone), and it mixes the  $\mu = \pm 1$  and  $\mu = 0$  eigenspaces of helicity. Similarly, the photon does not have a spin operator, because  $[\mathbf{S}, \Omega] \neq 0$  and because  $\mathbf{S}$  mixes the  $\mu = \pm 1$  and  $\mu = 0$  subspaces. On the other hand, the total angular momentum  $\mathbf{J} = \mathbf{L} + \mathbf{S}$  is meaningful as an operator acting on single photon states, since  $[\mathbf{J}, \Omega] = 0$ . Thus, it is possible to talk about the angular momentum states of a photon, it is just not possible to break this up into orbital and spin contribution as we can with a massive particle. (Indeed, as we will see when we study the Dirac equation, there is a more intimate coupling between spin and spatial degrees of freedom in relativistic quantum mechanics than in the nonrelativistic theory, even for massive particles such as the electron.)

Finally, there is one (Hilbert space) operator that does not commute with helicity but which nevertheless is defined for the photon, because it does not mix the  $\mu = \pm 1$  and  $\mu = 0$  subspaces. This is the parity  $\pi$ , defined in the usual way in nonrelativistic quantum mechanics, which satisfies

$$\pi \Omega \pi^\dagger = -\Omega \quad (62)$$

( $\pi$  flips the sign of  $\hat{\mathbf{k}}$ , but leaves  $\mathbf{S}$  alone). Parity maps the  $\mu = 1$  helicity subspace into the  $\mu = -1$  subspace (there is no mixing with  $\mu = 0$  states), and so is allowed for a photon.

Altogether, the single-particle operators that are or are not meaningful for a photon are summarized in Table 1. It may seem odd that helicity  $\Omega$  is a meaningful operator when it is defined in terms of  $\mathbf{S}$ , which is not meaningful; however, we can just as well write the helicity as  $\Omega = \hat{\mathbf{k}} \cdot \mathbf{J}$ , since  $\mathbf{k} \cdot \mathbf{L} = 0$ .

	Operators
Meaningful	$\Omega, \mathbf{k}, \mathbf{J}, \pi$
Not meaningful	$\mathbf{r}, \mathbf{L}, \mathbf{S}$

**Table 1.** Single particle operators that are defined for a massive particle are classified into those that are or are not meaningful for a massless particle, such as the photon.

Of course, once we have a photon wave function  $f(\mathbf{k}, \mu)$  in the  $(\mathbf{k}, \Omega)$ -representation, there is no harm in transforming it to another representation such as  $(\mathbf{r}, S_z)$ , as if it were the wave function of a massive particle, so long as we realize that only a restricted class of wave functions of  $(\mathbf{r}, m)$  will be allowed for a photon (namely, those lying in the  $\mu = \pm 1$  eigenspaces). To be explicit about

this, let us first specify the transformation between the  $(\mathbf{k}, \Omega)$ - and  $(\mathbf{k}, S_z)$ -representations; we will denote the respective wave functions by  $f(\mathbf{k}, \mu)$  and  $f(\mathbf{k}, m)$ . Then it is easy to show that

$$f(\mathbf{k}, m) = \sum_{\mu} D_{m\mu}^1(\hat{\mathbf{k}}) f(\mathbf{k}, \mu), \quad (63)$$

where  $D^1(\hat{\mathbf{k}})$  is the rotation matrix corresponding to the rotation  $R(\hat{\mathbf{k}})$  introduced in Eq. (30). Next, to transform from the  $(\mathbf{k}, S_z)$ -representation to the  $(\mathbf{r}, S_z)$ -representation, where we denote the wave function by  $f(\mathbf{r}, m)$ , we simply use the usual Fourier transform,

$$f(\mathbf{r}, m) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{r}} f(\mathbf{k}, m). \quad (64)$$

There is an alternative form for the wave functions  $f(\mathbf{k}, \mu)$ ,  $f(\mathbf{k}, m)$  or  $f(\mathbf{r}, m)$  of a spin-1 particle (massive or massless), in which the spin indices are replaced by Cartesian components of an ordinary 3-vector. This (Cartesian) form of the wave function can be specified in two equivalent forms,

$$\mathbf{f}(\mathbf{k}) = \sum_{\mu} \epsilon_{\mu}(\mathbf{k}) f(\mathbf{k}, \mu) = \sum_m \hat{\mathbf{e}}_m f(\mathbf{k}, m), \quad (65)$$

which is an ordinary (Cartesian) vector field over  $\mathbf{k}$ -space. The wave function  $f(\mathbf{k}, \mu)$  or  $f(\mathbf{k}, m)$  lies in the subspace spanned by the helicity states  $\mu = \pm 1$  if and only if  $\mathbf{f}(\mathbf{k})$  is transverse, that is,  $\mathbf{k} \cdot \mathbf{f}(\mathbf{k}) = 0$ . If we take the Fourier transform,

$$\mathbf{f}(\mathbf{r}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{r}} \mathbf{f}(\mathbf{k}), \quad (66)$$

we get a (Cartesian) vector field  $\mathbf{f}(\mathbf{r})$  in ordinary space that is missing the  $\mu = 0$  helicity state if and only if it is transverse, that is, if  $\nabla \cdot \mathbf{f}(\mathbf{r}) = 0$ . Such transverse, Cartesian vector fields  $\mathbf{f}(\mathbf{k})$  or  $\mathbf{f}(\mathbf{r})$  are often a convenient way of specifying the wave function of a photon.

We can now explain the absence of the (Hilbert space) operators  $\mathbf{r}$ ,  $\mathbf{L}$  and  $\mathbf{S}$  for a photon from another point of view. First, the orbital angular momentum  $\mathbf{L}$  is the generator of spatial rotations, which rotate the point of application of the wave function  $\mathbf{f}(\mathbf{r})$  or  $\mathbf{f}(\mathbf{k})$ . But if we rotate only the point of application  $\mathbf{k}$  and not the direction  $\mathbf{f}$  of the vector field itself, then the transversality condition  $\mathbf{k} \cdot \mathbf{f}(\mathbf{k}) = 0$  is not preserved. Similarly, the spin  $\mathbf{S}$  is the generator of rotations of the direction of the vector field  $\mathbf{f}$ , but not its point of application. This also does not preserve the transversality condition. Finally,  $\mathbf{r}$  is the generator of displacements in  $\mathbf{k}$ -space, and such displacements also do not preserve the transversality condition.

We will now make some comments about the various complete sets of commuting observables that are useful for free particle states, both for massive particles and for photons. In the case of a massive, spinless particle, the most obvious free particle wave functions are the momentum eigenfunctions  $\psi(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}$ , for which the CSCO is just  $\mathbf{k}$  (that is, the three commuting components of  $\mathbf{k}$ ). Such plane waves are not eigenstates of angular momentum, of course; if we desire these, then we can use the wave functions  $\psi(\mathbf{r}) = j_{\ell}(kr)Y_{\ell m}(\theta, \phi)$ , for which the CSCO is  $(k, L^2, L_z)$ . If

the particle has spin, then we can multiply by an eigenspinor of  $S_z$ , and obtain the wave functions  $\psi(\mathbf{r}, m) = e^{i\mathbf{k}\cdot\mathbf{r}}\chi_{m_s}^{m_s}$  or  $j_\ell(kr)Y_{\ell m}(\theta, \phi)\chi_{m_s}^{m_s}$ , for which the CSCO's are  $(\mathbf{k}, S_z)$  and  $(k, L^2, L_z, S_z)$ , respectively. (The spinor  $\chi_{m_s}^{m_s}$  has components  $\delta_{m, m_s}$  in the  $S_z$  basis, that is, it is an eigenspinor of  $S_z$  with quantum number  $m_s$ .) Finally, if we desire eigenstates of the total angular momentum, we can combine  $\mathbf{L}$  and  $\mathbf{S}$  with the Clebsch-Gordan coefficients for  $\ell \otimes s$ , to obtain the CSCO  $(k, L^2, J^2, J_z)$ .

None of these three obvious choices for the CSCO for the states of a massive free particle,  $(\mathbf{k}, S_z)$ ,  $(k, L^2, L_z, S_z)$ , or  $(k, L^2, J^2, J_z)$ , will work for a photon, because they all include one or more operators that are meaningless when the mass is zero. If we wish plane wave states, then we must replace  $S_z$  with something else. The helicity  $\Omega$  is convenient, and this leads to the plane wave, helicity eigenstates, for which the CSCO is  $(\mathbf{k}, \Omega)$ . These are the photon states created by our creation operators  $a_\mu^\dagger(\mathbf{k})$  [with the choice (33) for polarization vectors]. If we wish eigenstates of angular momentum, then we can include  $J^2$  and  $J_z$  in the CSCO, but we must replace  $L^2$  which may be used for a massive particle. It turns out there are two convenient substitutes for  $L^2$ , one being the helicity  $\Omega$ , and the other being parity  $\pi$ . Thus, we obtain two possible CSCO's for describing photons of definite angular momentum,  $(k, J^2, J_z, \Omega)$  and  $(k, J^2, J_z, \pi)$ . The latter choice is the more popular, because we are often interested in the conservation (or violation) of parity, as well as angular momentum. The single photon wave functions  $\mathbf{f}(\mathbf{r})$  which are simultaneous eigenfunctions of  $(k, J^2, J_z, \pi)$  are called the *vector multipole fields*, and are discussed in Jackson's book. They are messier to work with than plane waves, but necessary when a proper understanding of the conservation of angular momentum is desired.

At this point there are several topics for the free field that you should read about in Sakurai, *Advanced Quantum Mechanics*, including the statistical fluctuations in measurements of the quantized field, the criterion for validity of the classical description of the field, and the commutation relations among the fields at different space (or space and time) points.

We have one final topic to discuss regarding the free electromagnetic field, namely, statistical mechanics. This subject involves no new principles, but rather is a straightforward application of the usual rules of quantum statistical mechanics to the quantized field. In the following discussion, we will work with a box normalization, because this causes the various extensive properties of the system (energy, entropy, etc.) to be finite.

In practice, one often does not have exact knowledge of the state of the electromagnetic field, that is, the field is not in a pure state. Therefore it is necessary to describe the field by means of a density operator  $\rho$ , which is a nonnegative-definite operator of unit trace acting on  $\mathcal{E}_{\text{em}}$ , exactly as discussed in Notes 3.

The most important case in practice is that of thermal equilibrium, for which

$$\rho = \frac{1}{Z} e^{-\beta H_{\text{em}}}, \quad (67)$$

where we only include  $H_{\text{em}}$  in the Hamiltonian because we are only interested in the free field. As



usual, the normalization  $Z$  is the partition function,

$$Z(\beta) = \text{tr} e^{-\beta H_{\text{em}}}. \quad (68)$$

The trace is easily evaluated in the occupation number basis,

$$\begin{aligned} Z(\beta) &= \sum_{\{n_\lambda\}} \langle \dots n_\lambda \dots | \exp\left(-\beta \sum_\lambda \hbar\omega_k a_\lambda^\dagger a_\lambda\right) | \dots n_\lambda \dots \rangle \\ &= \sum_{\{n_\lambda\}} \exp\left(-\beta \sum_\lambda n_\lambda \hbar\omega_k\right) = \sum_{\{n_\lambda\}} \prod_\lambda e^{-n_\lambda \beta \hbar\omega_k}, \end{aligned} \quad (69)$$

where the sum is taken over all possible integer sequences  $\{n_\lambda\}$ . But the sum of products is the product of sums, so

$$Z(\beta) = \prod_\lambda \sum_{n_\lambda} e^{-\beta n_\lambda \hbar\omega_k} = \prod_\lambda \frac{1}{1 - e^{-\beta \hbar\omega_k}}, \quad (70)$$

where in the final expression we have summed the geometrical series.

From the partition function, any statistical average can be computed. For example, we easily find the average number of photons per mode in thermal equilibrium,

$$\langle n_\lambda \rangle = \frac{1}{e^{\beta \hbar\omega_k} - 1}, \quad (71)$$

which is the standard formula for the average occupation number in Bose-Einstein statistics of a single particle state in thermal equilibrium (massless particle only).