Physics 221A
Fall 2010
Appendix C
Gaussian Integrals

1. The Result

Consider the one-dimensional integral,

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2}, \tag{1}$$

where a is a complex number. If $\mathrm{Re}\, a > 0$, the integrand goes to zero exponentially as $x \to \pm\infty$, so the integral is defined. If $\mathrm{Re}\, a < 0$, the integrand diverges as $x \to \pm\infty$, so the integral is not defined. If $\mathrm{Re}\, a = 0$, so that a is purely imaginary, then the integrand oscillates with constant amplitude but shorter and shorter wave length as $x \to \pm\infty$, as long as $a \neq 0$. This is the marginal case, in which the integral still converges, since the areas of the positive and negative lobes forms a sequence with alternating signs whose terms approach zero. In summary, the integral (1) exists if either $\mathrm{Re}\, a > 0$, or $\mathrm{Re}\, a = 0$ and $a \neq 0$.

Under these conditions, write $a = re^{i\phi}$, where $-\pi/2 \le \phi \le \pi/2$ and where $r > 0$, and define $\sqrt{a} = \sqrt{r}\, e^{i\phi/2}$, so that $-\pi/4 \le \arg\sqrt{a} \le +\pi/4$. Then

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2} = \frac{\sqrt{\pi}}{\sqrt{a}}. \tag{2}$$

The most common cases in practice are those in which a is real and positive, for which

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}, \tag{3}$$

and the one in which a is purely imaginary, say, $a = -ic$ with c real and $c \neq 0$. In this case we have

$$\int_{-\infty}^{\infty} dx\, e^{icx^2} = e^{is\pi/4}\sqrt{\frac{\pi}{|c|}}, \tag{4}$$

where $s = \mathrm{sgn}\, c = \pm 1$. All these cases are included under (2), if $\sqrt{a}$ is properly understood.

A slightly more complicated case is the one-dimensional integral,

$$\int dx\, e^{-ax^2+bx} = \frac{\sqrt{\pi}}{\sqrt{a}}\, e^{b^2/4a}, \tag{5}$$

where the same conditions on a apply as above, where $\sqrt{a}$ is defined as above, and where b is any complex number.