Questions 1-4 deal with the situation sketched at right. Three identical charges, \( Q = -6 \, \mu\text{C} \), are placed at three corners of a square \( a = 0.10 \, \text{m} \) on a side.

1. (6 points) Calculate the magnitude of the electric field, in \( \text{N/C} \) or \( \text{V/m} \), at the fourth corner (point \( A \)).

Using the coordinate system indicated on the sketch:
\[ E = \frac{kQ}{a^2} \hat{y} + \frac{kQ}{a^2} \cos 135^\circ (\hat{x} \cos 135^\circ + \hat{y} \sin 135^\circ) - \frac{kQ}{a^2} \hat{x} \]

\[ = \frac{kQ}{a^2} \hat{y} + \frac{kQ}{2a^2} \left( -\frac{1}{\sqrt{2}} \hat{x} + \frac{1}{\sqrt{2}} \hat{y} \right) - \frac{kQ}{a^2} \hat{x} \]

\[ = -\frac{kQ}{a^2} (1 + \frac{1}{2\sqrt{2}}) \hat{x} + \frac{kQ}{a^2} (1 + \frac{1}{2\sqrt{2}}) \hat{y} \]

\[ |E| = \sqrt{E_x^2 + E_y^2} = \sqrt{\left[ \frac{kQ}{a^2} \left( 1 + \frac{1}{2\sqrt{2}} \right) \right]^2} = \frac{kQ}{a^2} \left( \sqrt{2} + \frac{1}{2} \right) \]

\[ = 1.0 \times 10^7 \text{ V m}^{-1}. \]

2. (2 points) T F If a positive charge is placed at point A and then released it will move towards point B at the center of the square.

3. (4 points) Calculate the electric potential, in V, at point A.

\[ V(A) = \frac{kQ}{a} + \frac{kQ}{a\sqrt{2}} + \frac{kQ}{a} = \frac{kQ}{a} \left( 2 + \frac{1}{\sqrt{2}} \right) \]

\[ = -1.5 \times 10^6 \text{ V.} \]

4. (6 points) Calculate the work done by an external force to move a charge \( q = +0.1 \mu\text{C} \) charge from point A to point B.

First calculate \( V(B) \):

\[ V(B) = \frac{kQ}{(a/\sqrt{2})} + \frac{kQ}{(a/\sqrt{2})} + \frac{kQ}{(a/\sqrt{2})} = \frac{kQ}{a} 3\sqrt{2} \]

\[ = -2.3 \times 10^6 \text{ V.} \]

Work is the change in potential energy, which is charge times the change in potential:

\[ W = U(B) - U(A) = q \left[ V(B) - V(A) \right] \]

\[ = -0.083 \text{ J.} \]

Questions 5-8 deal with the situation shown at right. A thin cylindrical shell of radius
$r_1 = 5.0 \text{ cm}$ is surrounded by a second cylindrical shell of a radius $r_2 = 10.0 \text{ cm}$. The inner shell has a charge per unit length of $\lambda_1 = +3.0 \mu\text{C/m}$ and the outer shell $\lambda_2 = -3.0 \mu\text{C/m}$. Assume the length $L$ of the shells to be much greater than the outer radius.

5. (3 points) Determine the magnitude of the electric field $E$ a distance $r = 3 \text{ cm}$ from the axis.

Since the charges are both cylindrically symmetric and essentially infinite, Gauss's Law can help here: $E$ must point radially and have uniform magnitude on any cylindrical Gaussian surface coaxial with the charges, and no flux will thread the circular ends of such a Gaussian surface.

So start with a Gaussian cylinder with radius $r = 3 \text{ cm}$ and length $\ell \ll L$. This lies completely inside the inner charged shell, and therefore encloses no charge:

$$\oint E \cdot dA = \frac{Q_{encl}}{\varepsilon_0}$$

$$E 2\pi r \ell = 0$$

$$E = 0$$

6. (6 points) Determine the magnitude of $E$ at $r = 8 \text{ cm}$.

A Gaussian cylinder with radius $r = 8 \text{ cm}$ and length $\ell \ll L$ encloses a length $\ell$ of the inner charge:

$$\oint E \cdot dA = \frac{Q_{encl}}{\varepsilon_0}$$

$$E 2\pi r \ell = \frac{\lambda_1 \ell}{\varepsilon_0}$$

$$E = \frac{\lambda_1}{2\pi\varepsilon_0 r} \hat{r} = \left(6.7 \times 10^5 \text{ V m}^{-1}\right) \hat{r}$$

7. (2 points) Determine the magnitude of $E$ at $r = 13 \text{ cm}$.

A Gaussian cylinder with radius $r = 13 \text{ cm}$ and length $\ell \ll L$ encloses a length $\ell$ of the each charge, making the total enclosed charge zero:
8. (3 points) A positive charge placed at \( r = 8 \text{ cm} \) is likely to move

a. toward the outer shell  

b. toward the inner shell  

c. clockwise

9. (3 points) A dielectric with dielectric constant \( K = 2 \) is inserted in a capacitor while it remains connected to the battery. The energy of the capacitor is

a. doubled  

b. quadrupled  

c. halved  

d. divided by 4

The dielectric doubles the capacitance, the voltage is constant, so \( U = CV^2/2 \) doubles.

10. (2 points) \( \boxed{\text{F}} \) It is possible to make two resistors of equal value with equal lengths of copper and aluminum wires.

\[ R = \rho \ell / A : \text{the aluminum wire would have larger cross-sectional area} \ A \text{ than the copper wire, as aluminum has larger resistivity than copper.} \]

11. (10 points \textbf{extra credit}) \textit{A two dimensional electric dipole.} Two infinite line charges lie parallel to each other and to the \( x \) axis, lie a distance \( a \) apart, and carry equal and opposite charges per unit length, \( \pm \lambda \). In the sketch at right these charges, and the \( x \) axis, are perpendicular to the page.

![Diagram of two line charges and the x-axis]

Derive a formula for the electric field \( E \) as a function of \( z \), the axis perpendicular to the line charges with origin between them.

Since the charges are infinite and straight, the fields of each can be determined from Gauss's Law and then superposed. Surround, for instance, a length \( \ell \) of the upper
charge with a coaxial Gaussian cylinder of radius \( r' \) and length \( \ell \); then, as we found above, the electric field is

\[
E_U = \frac{\lambda}{2\pi\varepsilon_0 r'_U} \hat{r}'_U ,
\]

where \( r'_U \) is measured from the charge, and \( \hat{r}'_U \) points away from the charge (not the coordinate origin). The \( z \) component of the field, which is all we're asked about, is

\[
E_{Uz} = \frac{\lambda}{2\pi\varepsilon_0 z_U} = \frac{\lambda}{2\pi\varepsilon_0 (z - a/2)} .
\]

Similarly, the \( z \) component of the field from the lower charge is

\[
E_{Lz} = \frac{-\lambda}{2\pi\varepsilon_0 (z + a/2)} .
\]

So the \( z \) component of the "total" field is

\[
E_z = E_{Uz} + E_{Lz} = \frac{\lambda}{2\pi\varepsilon_0} \left( \frac{1}{z - a/2} - \frac{1}{z + a/2} \right) = \frac{\lambda}{2\pi\varepsilon_0} \left( \frac{z + a/2 - z + a/2}{z^2 - (a/2)^2} \right) = \frac{\lambda a}{2\pi\varepsilon_0} \frac{1}{z^2 - (a/2)^2} .
\]

Note that at very large distances from the charges (\( z \gg a \)), the electric field decreases with increasing distance according to \( 1/z^2 \), different from the \( 1/z^3 \) dependence we get along the axis of ordinary electric dipoles in 3-D.

Questions 12-14. An \( \alpha \)-particle is moving at a speed of \( v = 5 \times 10^5 \) m s\(^{-1}\) in the direction perpendicular to a uniform magnetic field of magnitude \( B = 4 \times 10^{-2} \) T. The charge on an \( \alpha \)-particle is \( Q = 3.2 \times 10^{-19} \) C and its mass is \( m = 6.6 \times 10^{-27} \) kg.

12. (2 points) As the \( \alpha \)-particle moves through magnetic field its kinetic energy

a. decreases    b. increases    c. stays the same    d. depends on the direction of the magnetic field

Magnetic forces do no work.
13. (4 points) What is the magnitude of the magnetic force on the α-particle?

\[ F = QvB \sin 90^\circ = QvB = 6.4 \times 10^{-15} \text{ N.} \]

14. (6 points) What is the time it takes the α-particle to complete one full revolution around its path?

The α-particle executes cyclotron motion, moving in a circle with radius \( r \) determined from

\[ F = QvB = ma = \frac{mv^2}{r} \quad \Rightarrow \quad r = \frac{mv}{QB} \]

It travels with unchanged speed \( v \) in that circle, so a complete revolution takes

\[ \tau = \frac{2\pi r}{v} = \frac{2\pi m}{QB} = 3.2 \times 10^{-6} \text{ s.} \]

15. (3 points) What is the direction of the magnetic force on the wire shown at right?

a. ↑  b. ↓  c. ←  d. →  e. out (⊕)  f. in (⊗)

16. (2 points) What is not an example of electromagnetic waves?

a. Light  b. Radio waves  c. Sound  d. X-rays

17. (2 points) T F In electromagnetic waves, electric field is parallel to magnetic field.

18. (2 points) T F If rod type antennas are used for both transmission and reception of radio signal, the best reception can be obtained when they are parallel.

19. (8 points) A very long wire which carries a current \( I \) is straight except for a semicircular loop with radius \( a \) centered at point \( P \), as shown in the sketch at right. Derive a formula for the magnetic field \( B \) (magnitude and direction) at point \( P \).
A sensible coordinate system would be centered at $P$ with the $x$ axis running rightward along the wire, as shown; then $r = 0$ and $\mathbf{r} - \mathbf{r}' = -\mathbf{r}' = -\hat{r}$. Break the problem into two parts: the straight segments and the semicircle. The straight segments don't contribute to the magnetic field at $P$ because

$$dB = \frac{\mu_0}{4\pi} \frac{Idl \times (\mathbf{r} - \mathbf{r}')}{|r - r'|^2} = \frac{\mu_0}{4\pi} \frac{-I dx \hat{x} \times (\mp \hat{x})}{x^2} = 0,$$

whether the current element lies at $x > a$ (for which $\mathbf{r} - \mathbf{r}' = -\hat{x}$) or $< -a$ ($\mathbf{r} - \mathbf{r}' = +\hat{x}$).

For the semicircle, the appropriate current element is $Idl = lad \phi \hat{r}$, and $\phi$ runs from zero to $\pi$. Thus

$$B(P) = \frac{\mu_0}{4\pi} Idl \times (\mathbf{r} - \mathbf{r}') = \frac{\mu_0}{4\pi} \int_0^\pi lad \phi \times (-\hat{r}) = \frac{\mu_0 I}{4\pi a} \int_0^\pi d\phi = \frac{\mu_0 I}{4a} \hat{z}.$$

$\hat{z}$ points out of the page.

20. (2 points) Calculate $B$ at point $P$ for $a = 1$ cm and $I = 1$ A.

$$B(P) = \frac{\mu_0 I}{4a} \hat{z} = \left(3.1 \times 10^{-5} \text{ T}\right) \hat{z}.$$

21. (4 points) Another long wire, carrying the same current $I$, is straight except for a rectangular loop with dimensions $a \times 2a$ around point $O$, as shown at right. Compared to the magnetic field at point $P$ in the wire considered above, the field at point $O$ is

a. zero.  b. smaller.  c. the same.  d. larger.

**Questions 22 – 26** deal with this $LR$ circuit which has $L = 1$ mH and $R = 100$ $\Omega$. The circuit is connected to an AC generator with amplitude $V_0 = 10$ V.

22. (6 points) Obtain formulas for the magnitude
and phase of the impedance of the circuit.

Series combination:

\[ Z = i\omega L + R = Z_0 e^{i\phi} \]

\[ Z_0 = \sqrt{(\text{Re} Z)^2 + (\text{Im} Z)^2} = \sqrt{R^2 + (\omega L)^2} = R\sqrt{1 + (\omega L/R)^2} \]

\[ \phi = \arctan \left( \frac{\text{Im} Z}{\text{Re} Z} \right) = \arctan \left( \frac{\omega L}{R} \right) \]

23. (6 points) Obtain a formula for the voltage across the resistor, \( V_{\text{out}} \).

\[ I = \frac{V}{Z} = \frac{V_0 e^{i\omega t}}{Z_0 e^{i\phi}} = \frac{V_0}{R\sqrt{1 + (\omega L/R)^2}} e^{i(\omega t - \phi)} \]

\[ V_{\text{out}} = IR = \frac{V_0}{\sqrt{1 + (\omega L/R)^2}} e^{i(\omega t - \phi)} \]

\[ \text{Re} V_{\text{out}} = \frac{V_0}{\sqrt{1 + (\omega L/R)^2}} \cos \left( \omega t - \arctan \left[ \frac{\omega L}{R} \right] \right) \]

24. (6 points) Obtain a formula for the frequency, \( f_{1/2} \), at which the amplitude of \( V_{\text{out}} \) equal to half of \( V_0 \), and calculate the numerical value of this frequency in Hz.

\[ \frac{(V_{\text{out}})_0}{V_0} = \frac{1}{\sqrt{1 + (\omega L/R)^2}} = \frac{1}{2} \]

\[ 1 + (\omega_{1/2}/R)^2 = 4 \]

\[ \omega_{1/2} = \frac{R\sqrt{3}}{L} = 2\pi f_{1/2} \quad \Rightarrow \quad f_{1/2} = \frac{\sqrt{3} R}{2\pi L} = 28 \text{ kHz} \]

25. (2 points) At frequencies much larger than \( f_{1/2} \), the amplitude of \( V_{\text{out}} \) is

a. much smaller than \( V_0 \).  
   b. the same as \( V_0 \).  
   c. much larger than \( V_0 \).

26. (2 points) At frequencies much smaller than \( f_{1/2} \), the amplitude of \( V_{\text{out}} \) is

a. much smaller than \( V_0 \).  
   b. the same as \( V_0 \).  
   c. much larger than \( V_0 \).

In other words, this circuit is a low-pass filter.
27. (10 points extra credit) The center of a square loop (side \(a = 5.0 \text{ cm}\)) is located at a distance \(r = 10 \text{ cm}\) from a wire carrying current \(I_{\text{wire}}\) parallel to one of the sides of the loop. The loop is made of copper wire (resistivity \(\rho = 1.7 \times 10^{-8} \Omega \text{ m}\)) with cross-sectional area \(A = 4 \text{ mm}^2\). At \(t = 0\), the current \(I\) in the long wire starts increasing linearly, at the rate \(\frac{dI}{dt} = 1.5 \text{ A s}^{-1}\). Determine the current \(i\) in the square loop at \(t = 0.5 \text{ s}\).

This is a Faraday's Law problem: we first have to work out the magnetic field, then the magnetic flux and its time derivative, then the induced current.

The resistance of the square loop is

\[
R = \frac{\rho d}{A} = \frac{4 \rho a}{A} = 1.7 \times 10^{-4} \Omega.
\]

If we point the \(z\) axis of a coordinate system along the wire in the direction of the current, the magnetic field generated by this current is obtained by Ampère's Law with a circular loop:

\[
\oint B \cdot dl = \mu_0 I_{\text{encl}}
\]

\[
B2\pi r = \mu_0 I \quad \Rightarrow \quad B(r) = \frac{\mu_0 I}{2\pi r} \hat{\phi}
\]

which points into the page to the right of the wire, where the loop is.

The flux through the loop should be worked out by integration since it is not far away from the wire compared to its own dimensions. Take \(dA = adr'\), since the field is the same at all \(z\) for a given distance from the wire:

\[
\Phi_B = \int B \cdot dA = \int_{r-a/2}^{r+a/2} \left( \frac{\mu_0 I}{2\pi r'} \right) (adr') = \frac{\mu_0 Ia}{2\pi} \ln \frac{r+a/2}{r-a/2}
\]

\[
= \frac{\mu_0 Ia}{2\pi} \ln \frac{r+a/2}{r-a/2}
\]

Everything on the right-hand side is constant except for the current \(I\), so the induced EMF and current are
The minus sign indicates that the current flows in the direction that opposes the flux change: counterclockwise, since the increase in $I$ leads to more flux into the page and a counterclockwise current creates a magnetic field which points out of the page for points within the loop.

It's not as inaccurate as one might think, though, to assume the external field threading the loop to be uniform at its central value. If one makes this approximation, the integral is unnecessary, and the answers turn out to be

\[
\Phi_B = \mu_0 I a^2 / 2 \pi r \quad \text{and} \quad i = -(dI/dt) \mu_0 a^2 / 2 \pi r R = -4.4 \times 10^{-5} \ \text{A}.
\]