Today in Physics 122: review of DC circuits, magnetostatics, and induction

Shanghai’s high-speed maglev train, leaving the airport (Shanghai Metro).
The second midterm exam

Midterm #2: 8-9:15 AM, **Tomorrow**, 13 November 2012, Hubbell Auditorium (141 Hutchison Hall)

- Covers all topics reviewed here: those introduced since the last exam (chapters 26-29 in the book)
- Please bring **only** a calculator and a writing instrument to the exam.
DC circuits: combinations of $R_s$, $C_s$

Parallel and series pairs of resistors and capacitors:

\[
\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} \\
C_{eq} = \frac{C_1 C_2}{C_1 + C_2} < C_1, C_2
\]

\[
\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \\
R_{eq} = \frac{R_1 R_2}{R_1 + R_2} < R_1, R_2
\]
DC circuits: Kirchhoff’s Rules

1. At each node – the conductor of whatever shape that joins two or more components – the current flowing in is equal to the current flowing out.

2. Voltages along any complete loop in a circuit must add up to zero.

\[
I_1 - I_2 - \frac{dQ}{dt} = 0
\]
\[
V_0 - I_1 R_1 - I_2 R_2 = 0
\]
\[
I_2 R_2 - \frac{Q}{C} = 0
\]
Kirchhoff’s Rule recipe

- Define a current – value and direction – through each component in the circuit.
  - Current \(\frac{dQ}{dt}\) flows into + end of C, from + end of \(V\), and +\(\rightarrow\)- in \(R\).

- In the node rule (KR#1), bookkeep all currents flowing into the node as positive, and all those flowing out as negative; the sum of all the positives and negatives comes to zero.
Kirchhoff’s Rules recipe (continued)

- In the loop rule (KR#2), count voltages traversed from - to + as positive, and those traversed from + to - as negative.
  - That is, voltage **drops** across a resistor if you’re following voltage in the direction the current flows through it.

\[ I_1 - I_2 - \frac{dQ}{dt} = 0 \]
\[ V_0 - I_1 R_1 - I_2 R_2 = 0 \]
\[ I_2 R_2 - \frac{Q}{C} = 0 \]
Kirchhoff’s Rules recipe (continued)

- Identify the unknown quantities – $N$, say – in the circuit, and count them.
- Then write the node rule and/or the loop rule to generate as many relations between the voltages and currents as there are unknowns ($N$).
  - Use both the node loop rules, at least once each.
- This gives a system of $N$ equations in $N$ unknowns, either purely linear equations (if only $R$ and $V$), or first-order differential equations (if there are Cs). Solve them.
- Important point: it doesn’t matter whether you correctly guess the direction of each current. If you guess wrong, your answer will just be a negative number.
Magnetostatics

Force laws:
\[ dF = I dl \times B = I dl B \sin \theta \hat{n} \]
\[ F = \int I dl \times B = \int I B \sin \theta \hat{n} dl \]
\[ F = Q v \times B \]  
point charges

Cross products:
\[ a \times b = -b \times a = ab \sin \theta \hat{n} \]
\[ : a \times b = 0 \text{ if } a \parallel b \]
\[ : a \times b = ab \hat{n} \text{ if } a \perp b \]

Cross products of Cartesian unit vectors:
\[ \hat{x} \times \hat{y} = \hat{z} \]
\[ \hat{y} \times \hat{z} = \hat{x} \]
\[ \hat{z} \times \hat{x} = \hat{y} \]

Lay fingers of right hand along arc from \( dl \) to \( B \), pointing to \( B \).
Thumb points along \( dF \).
Magnetostatics (continued)

Field laws:

\[ B = \frac{\mu_0}{4\pi} \int \frac{Id\ell \times (r-r')}{|r-r'|^2} \]

Biot-Savart field law

\[ \oint_C B \cdot d\ell = \mu_0 I_{\text{enc}} \]

\[ = \mu_0 \int_A J \cdot dA \]

where \( \mu_0 = 4\pi \times 10^{-7} \text{ T m A}^{-1} \)

Or: thumb along \( Id\ell \); fingers curl in direction of \( dB \).
Magnetostatics (continued)

Field laws:

\[ B = \frac{\mu_0}{4\pi} \int \frac{I d\ell \times (\vec{r} - \vec{r}')}{|r - r'|^2} \]

\[ \oint_C B \cdot d\ell = \mu_0 I_{\text{encl}} \]

\[ = \mu_0 \int_A J \cdot dA \]

where \( \mu_0 = 4\pi \times 10^{-7} \text{ T m A}^{-1} \)
Biot-Savart field law recipe

The setup, and most of the execution, of $B$ calculations from the Biot-Savart field law are the same as for $E$ calculations using Coulomb’s law.

That is,

- choose an appropriate coordinate system,
- dissect the source distribution into infinitesimal elements,
- use the symmetry of the source distribution to simplify the vector addition as much as possible, and then
- integrate the resulting expression.

No new tricks are involved, and no new complications besides the intrusion of cross products, and fields that lie sideways with respect to the distance from the infinitesimal element.
Ampère’s Law recipe

Though the dimension of its integration is reduced by one on both sides of the equation compared to Gauss’s Law, Ampère’s Law is applied to find $B$ in much the same way that Gauss’s Law is applied to find $E$.

That is, use the symmetry of the current distribution to

- determine the direction of $B$, and
- identify closed curves along which the magnitude of $B$ is uniform, or along which

- Then $B$ comes out of the integral; the two remaining integrals are simple, enabling solution for $B$.

<table>
<thead>
<tr>
<th>Ampère’s law geometries</th>
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<tbody>
<tr>
<td>Infinite linear current</td>
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<tr>
<td>Infinite planar current</td>
</tr>
<tr>
<td>Infinite cylindrical current, any radial dependence</td>
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<tr>
<td>Infinite solenoid</td>
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<tr>
<td>Toroid</td>
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Magnetic materials

Permeability: $\mu = K_m \mu_0$.

Diamagnetic ($K_m < 1$; e.g. Cu) and paramagnetic ($K_m > 1$; e.g. Al) materials have small permeability ($|K_m - 1| = 10^{-5} - 10^{-4}$). Ferromagnetic materials (e.g. Fe, Ni, Co, Sm) can have very large permeability: $K_m \sim 1000$. 

$B = \mu n I = K_m \mu_0 n I$

$K_m > 1$ as drawn.
Induction and Faraday’s Law

- If the flux of $B$ enclosed by a circuit changes with time, an electromotive force is induced, that drives a current and associated additional $B$, which opposes the change.

- Equivalently: if the flux of $B$ through a surface changes with time, so does the integral of $E$ around the boundary of the surface.

Stated instead as equations,

\[
\mathcal{E} = - \frac{d\Phi_B}{dt} = \oint_C E \cdot dl = - \frac{d}{dt} \int_S B \cdot dA
\]

Faraday’s Law
Faraday’s Law recipe

- Calculate the magnetic flux, and differentiate to get its rate of change. This tells you the magnitude of the induced EMF right away.

- To get the polarity, consider how the EMF can produce a current, and thus its own $B$, to oppose the change in flux.
  - If $\int B \cdot dA$ is decreasing, make $I$ produce more field in the same direction as the external field.
  - If $\int B \cdot dA$ is increasing, make $I$ produce more field in the opposite direction as the external field.
  - Remember which way $I$ produces $B$. (See figure.)
General advice

Don’t forget the chain rule of differentiation or the fundamental theorem of calculus.

- For example, if you know \( \Phi_B(x) \) and want to know \( d\Phi_B/dt \), note that
  \[
  \frac{d\Phi_B}{dt} = \frac{d\Phi_B}{dx} \frac{dx}{dt} = \frac{d\Phi_B}{dx} v_x.
  \]

- Or: if you know you’re going to have to take the derivative of a function you just produced by integration, save a step with
  \[
  \frac{d}{dx} \int_a^x f(x') \, dx' = f(x).
  \]
Don’t forget the principle of superposition: a complicated current, for example, can often be broken down into two simple ones for which the fields are calculated simply and then superposed. Like uniform current in a cylinder with an off-center hole:

\[ J = \frac{I}{A} \]

\[ J + -J \]

Don’t forget the cross products of Cartesian unit vectors (page 7).
One more example

What will you have?
2. Tension in a current loop immersed in a magnetic field.
3. $B$ from Biot-Savart law.
4. $B$ from Ampère’s law.
5. Induction and force.