Today in Physics 122: AC circuit examples

- Q&A on Midterm #2 (if desired)
- Resonant circuits: series and parallel $LRC$
- Filters: narrowband, lowpass, highpass
Reminders about AC circuits

If an electrical circuit with $L_s$, $R_s$ and $C_s$ in it is driven by a sinusoidal voltage,

- said voltage can be represented for purposes of algebra by a complex voltage, of which we **only measure the real part**:

\[
V = V_0 e^{i(\omega t + \phi)}
\]

\[
= V_0 \cos(\omega t + \phi) + i \sin(\omega t + \phi)
\]

\[
\text{Re}(V) = V_0 \cos(\omega t + \phi)
\]

- Such complex terms can be represented in vector-like fashion in a phasor diagram, as at right.
Reminders about AC circuits (continued)

- Circuit components: inductors, resistors, and capacitors can be represented by reactances: complex forms of resistance.
  
  \[ X_L = i\omega L = \omega Le^{i\pi/2} \]
  \[ X_R = R \]
  \[ X_C = \frac{1}{i\omega C} = \frac{1}{\omega C}e^{-i\pi/2} \]

- Reactances add in series and parallel just like resistors do, except for the complex-number arithmetic required. The equivalent reactance of a network of components is the **impedance**.
Reminders about AC circuits (continued)

- Algebra on the reactances, e.g.
  \[ Z = i\omega L + R + \frac{1}{i\omega C} \]
  is equivalent to vector addition of phasors, as at right.

- The current through a complex impedance \( Z \) “driven” by a sinusoidal voltage \( V \) is represented by a complex current,
  \[ I = \frac{V}{Z}, \]
  of which only the real part is measured.
Reminders about imaginary and complex numbers

\[ i = \sqrt{-1} = e^{i\pi/2} \]

\[ -i = e^{-i\pi/2} = \frac{1}{i} \]

\[ Z = Ae^{i\theta} = A \cos \theta + iA \sin \theta \]

\[ A = \sqrt{\text{Re} Z^2 + \text{Im} Z^2} \]

\[ \theta = \arctan \left( \frac{\text{Im} Z}{\text{Re} Z} \right) \]

Electrical engineers usually refer to \( i \) as \( j \), for perverse reasons best known to themselves.
Parallel LRC circuit

Problem 30-93 in the book, except for a 90° phase shift:

Determine the current through each component in this parallel LRC circuit, and the total current leaving the voltage source.
Parallel LRC circuit (continued)

Individual currents, \( I = V_0 e^{i\omega t} / X \):

\[
I_L = \frac{V_0}{i\omega L} e^{i\omega t} \\
I_R = \frac{V_0}{R} e^{i\omega t} \\
I_C = i\omega CV_0 e^{i\omega t} = \omega CV_0 e^{i(\omega t + \pi/2)}
\]

for a total of

\[
I = \frac{V_0}{R} e^{i\omega t} + \frac{V_0}{i\omega L} e^{i\omega t} + i\omega CV_0 e^{i\omega t} = \frac{V_0 e^{i\omega t}}{R} \left( 1 + i \left[ \omega RC - \frac{R}{\omega L} \right] \right)
\]
Put term in brackets into the form $A e^{i\phi}$:

$$\tilde{A} = 1 + i\left(\omega RC - \frac{R}{\omega L}\right)$$

$$A = \sqrt{\left(\text{Re}\tilde{A}\right)^2 + \left(\text{Im}\tilde{A}\right)^2} = \left(1 + R^2 \left[\frac{\omega^2 LC - 1}{\omega L}\right]^2\right)^{1/2}$$

$$\phi = \arctan\left(\frac{\text{Im}\tilde{A}}{\text{Re}\tilde{A}}\right) = \arctan\left(R\left[\frac{\omega^2 LC - 1}{\omega L}\right]\right)$$
Parallel LRC circuit (continued)

... and the total current $I$ becomes

$$I = \frac{V_0}{R} \left(1 + R^2 \left[\frac{\omega^2LC - 1}{\omega L}\right]^2\right)^{1/2} e^{i(\omega t + \phi)} = I_0 e^{i(\omega t + \phi)}$$

with $\phi$ as given on the previous page.

Only the real part of the current gets measured, so the final answer is $I = I_0 \cos(\omega t + \phi)$, with $I_0$ and $\phi$ as given above.
Parallel LRC circuit (continued)

The circuit draws a lot of current except for angular frequencies near $\omega = \sqrt{1/LC}$; the resonance in this case is a maximum of impedance.

(The series LRC circuit has a minimum of impedance at resonance, as we’ll see in a minute.)
Problem 30-91a and b in the book, except for a phase shift: 

Determine the current through the capacitor, if $X_L > X_C$, and determine the amplitude of the voltage across the capacitor.

$$V_{in} = V_0 \cos \omega t$$
Narrowband filter (continued)

\[ V_{in} = V_0 \cos \omega t \]

The current through the capacitor is the same as that through the inductor, as they’re in series:

\[ Z = X_L + X_C = i\omega L + \frac{1}{i\omega C} = i \left( \frac{\omega^2 LC - 1}{\omega L} \right) \]

\[ = 0 \quad \text{for} \quad \omega = \sqrt{1/LC} \]

(A bit pathological, since the resistance of a real circuit wouldn’t be zero.)
Narrowband filter (continued)

At any rate, this leads to a sharp peak in the current at the resonance frequency:

\[
I = \frac{V_{\text{in}}}{Z} = \frac{V_0 e^{i\omega t}}{i(\omega L - 1/\omega C)} = \frac{\omega CV_0 e^{i(\omega t - \pi/2)}}{\omega^2 LC - 1}
\]

\[
V_{\text{out}} = IX_C = \frac{1}{i\omega C} \frac{\omega CV_0 e^{i(\omega t - \pi/2)}}{\omega^2 LC - 1} = -\frac{V_0 e^{i\omega t}}{\omega^2 LC - 1}
\]
Narrowband filter (continued)

The amplitude:

\[
(V_{\text{out}})_0 = |V_{\text{out}}| = \frac{V_0}{\omega^2 LC - 1}
\]

This is a useful building block for signal-processing circuits: only voltages with frequencies near the resonance frequency are transmitted from the input to the output.
Low-pass filter

Problem 30-103 in the book:

Find the **voltage gain**, $V_{out}/V_{in}$, in this circuit, and discuss the behavior of the gain when $f \to 0$ and $f \to \infty$.

\[
V_{in} = V_0 \cos \omega t
\]

\[
V_{out}
\]
Low-pass filter (continued)

\[ V_{\text{in}} = V_0 \cos \omega t \]

As before, we compute the impedance and current:

\[ Z = X_R + X_C = R + \frac{1}{i\omega C} = \frac{\omega RC - i}{\omega C} \]

\[ I = \frac{V_{\text{in}}}{Z} = \frac{\omega CV_0 e^{i\omega t}}{\omega RC - i} = \frac{\omega CV_0 (\omega RC + i) e^{i\omega t}}{(\omega RC)^2 + 1} \]

\[ V_{\text{out}} = IX_C = \frac{V_0 (1 - i\omega RC) e^{i\omega t}}{(\omega RC)^2 + 1} \]
Low-pass filter (continued)

Express in the form $A e^{i\phi}$:

$$V_{\text{out}} = \frac{V_0 e^{i\omega t}}{1 + (\omega RC)^2} \sqrt{1 + (\omega RC)^2} e^{i \arctan(-\omega RC)}$$

$$= \frac{V_0}{\sqrt{1 + (\omega RC)^2}} e^{i(\omega t - \arctan[\omega RC])}$$

Again, recall that only the real part is measured.
The voltage gain is the ratio of the input and output voltages:

\[ g = \text{Re} \left( \frac{V_{\text{out}}}{V_0 e^{i\omega t}} \right) = \frac{1}{\sqrt{1 + (\omega RC)^2}} \cos \left( \arctan(\omega RC) \right) \]

\[ = \frac{1}{\sqrt{1 + (\omega RC)^2}} \left( \frac{1}{\sqrt{1 + (\omega RC)^2}} \right) = \frac{1}{1 + (\omega RC)^2} \]

If \( \tan \theta = a/b \), then \( \cos \theta = b/\sqrt{a^2 + b^2} \).
Note the limits of low and high frequency:

\[ g(\omega) = \frac{1}{1 + (\omega RC)^2} \]

\[ g(0) = 1 \]

\[ g(1/RC) = \frac{1}{2} \]

\[ \lim_{\omega \to \infty} g(\omega) = 0 \]

The circuit efficiently transmits angular frequencies below \(1/RC\), and attenuates those above this value; hence its name.
Single-pole lowpass and highpass filters

Lowpass

\[ V_{in} \xrightarrow{R} V_{out} \]

Highpass

\[ V_{in} \xrightarrow{L} V_{out} \]

\[ V_{in} \xrightarrow{C, R} V_{out} \]