Today in Physics 122

- One more complicated electric-field example
- Small and infinite – or at least very large – charge distributions
- How to make a uniform $E$ field
- Dipoles
- Torque on a dipole in a uniform electric field.

\[ E = E\hat{y} \]
Example: field of a line charge

A straight wire of length $L$ carries a total charge $Q$. Calculate the electric field everywhere.

- Straight wires are one-dimensional:
  \[ \lambda = \frac{Q}{L} \]

- Better run one axis along the wire. It doesn’t really matter where along the wire the origin goes, but the midpoint will turn out to be good.
Field of a line charge (continued)

- The obvious choice for an infinitesimal element of length is $dx'$, making $dQ = \lambda dx'$. Then the integral can run from $x' = -L/2$ to $L/2$.

- The distance to the test point from $x'$ is such that

$$ (r - r')^2 = (x - x')^2 + y^2 $$
Field of a line charge (continued)

From the triangle formed by the axes and $r - r'$,

$$\sin \theta = \frac{y}{\sqrt{(x-x')^2 + y^2}}$$

$$\cos \theta = \frac{x-x'}{\sqrt{(x-x')^2 + y^2}}$$

Now we write Coulomb's law for the field, one component at a time:
Field of a line charge (continued)

\[
E_x = \int \kappa \frac{dQ}{(r-r')^2} \cos \theta = \int_{-L/2}^{L/2} \kappa \frac{\lambda dx'}{(x-x')^2 + y^2} \frac{x-x'}{\sqrt{(x-x')^2 + y^2}}
\]

- With a minor variable substitution,
  \[X = x - x' \implies dX = -dx'
  \]
  \[x' = -L/2 \rightarrow L/2 \implies X = x + L/2 \rightarrow x - L/2\]

we see this is actually the integral we did with the charged disk:

\[
E_x = -\kappa \lambda \int_{x+L/2}^{x-L/2} \frac{X dX}{\left(X^2 + y^2\right)^{3/2}} = \kappa \lambda \left[ \frac{1}{\sqrt{X^2 + y^2}} \right]_{x+L/2}^{x-L/2}
\]
Field of a line charge (continued)

or

\[ E_x = k\lambda \left[ \frac{1}{\sqrt{(x-L/2)^2 + y^2}} - \frac{1}{\sqrt{(x+L/2)^2 + y^2}} \right] \]

Now the \( y \) component:

\[ E_y = \int \frac{kdQ}{(r-r')^2} \sin \theta = k\lambda y \int_{-L/2}^{L/2} \frac{dx'}{\left((x-x')^2 + y^2\right)^{3/2}} \]

In the spirit of the textbook, we can look up the integral in a table:

\[ E_y = -\frac{k\lambda}{y} \left[ \frac{x-x'}{\sqrt{(x-x')^2 + y^2}} \right]_{-L/2}^{L/2} \]
Optional: it’s not that hard to integrate one’s self: make a trigonometric substitution:

\[ x - x' = y \tan \alpha \quad \Rightarrow \]

\[ -dx' = y \left(1 + \tan^2 \alpha\right) d\alpha = \frac{y}{\cos^2 \alpha} \, d\alpha \]

\[ x' = -L/2 \rightarrow L/2 \quad \Rightarrow \]

\[ \alpha = \arctan \left(\frac{x + L/2}{y}\right) \rightarrow \arctan \left(\frac{x - L/2}{y}\right) = \alpha_+ \rightarrow \alpha_- \]

The denominator under the integral becomes

\[ \left( (x - x')^2 + y^2 \right)^{3/2} = \left( y^2 \tan^2 \alpha + y^2 \right)^{3/2} = \frac{y^3}{\cos^3 \alpha} \]
so \( E_y = -k\lambda y \int_{\alpha_-}^{\alpha_+} \frac{\cos^3 \alpha}{y^3} \frac{y d\alpha}{\cos^2 \alpha} = + \frac{k\lambda}{y} \int_{\alpha_-}^{\alpha_+} \cos \alpha d\alpha = \frac{k\lambda}{y^2} \sin \alpha \bigg|_{\alpha_-}^{\alpha_+} \)

\[
= \frac{k\lambda}{y} \left[ \sin \left( \arctan \left( \frac{x + L/2}{y} \right) \right) - \sin \left( \arctan \left( \frac{x - L/2}{y} \right) \right) \right]
\]

\[
= \frac{k\lambda}{y} \left[ \frac{x + L/2}{\sqrt{(x + L/2)^2 + y^2}} - \frac{x - L/2}{\sqrt{(x - L/2)^2 + y^2}} \right]
\]

(Recall that if \( \tan \theta = a/b \), then \( \sin \theta = a/\sqrt{a^2 + b^2} \) and \( \cos \theta = b/\sqrt{a^2 + b^2} \).)
Field of a line charge (continued)

So,

\[
E_y = \frac{k\lambda}{y} \left[ \frac{x + L/2}{\sqrt{(x + L/2)^2 + y^2}} - \frac{x - L/2}{\sqrt{(x - L/2)^2 + y^2}} \right]
\]

Note that if \( x = 0 \), \( E_x = 0 \), and thus

\[
E = \frac{k\lambda}{y} \frac{L}{\sqrt{(L/2)^2 + y^2}} \hat{y}
\]

as you might have expected from the symmetry.
Other useful consequences of our recent Examples

1. Far away from a line charge

If the length $L$ is very small compared to $y$, then the field in the plane through the midpoint looks even simpler:

$$E(0, y) = k \frac{\lambda L}{y \sqrt{(L/2)^2 + y^2}} \hat{y} \approx k \frac{Q}{y^2} \hat{y}$$

just as it’s supposed to from a place far enough away that the line looks like a point (Coulomb’s law).

Examining a limit in which you know what the answer is, is a good way of checking your answer.
Other useful consequences of our recent Examples (continued)

2. Close to a line charge (or anywhere, in the presence of an infinitely-long line charge)

If on the other hand \( L \) is much larger than \( y \),

\[
E(0, y) = k \frac{\lambda L}{y \sqrt{(L/2)^2 + y^2}} \hat{y} \approx k \frac{2\lambda \hat{y}}{y}
\]

neglect, compared to \((L/2)^2\)

\( E \) points perpendicularly away from a long line charge, and its magnitude drops off with increasing distance as \( 1/y \).
3. Far from a charged disk (optional).

For those familiar with the binomial approximation, \((1 + x)^n \approx 1 + nx\) if \(|x| \ll 1\):

\[
E(z) = 2\pi k\sigma\hat{z} \left(1 - \frac{z}{\sqrt{z^2 + R^2}}\right) = 2\pi k\sigma\hat{z} \left(1 - \frac{1}{\sqrt{1+(R/z)^2}}\right)
\]

\[
\approx 2\pi k\sigma\hat{z} \left(1 - \left[1 - \frac{1}{2} \left(\frac{R}{z}\right)^2\right]\right) = k \frac{\pi R^2 \sigma}{z^2} \hat{z}
\]

\[
= k \frac{Q}{z^2} \hat{z}. \quad \text{again recovering Coulomb's law.}
\]
Other useful consequences of our recent Examples (continued)

4. Near a charged disk, or in the presence of an infinite charged disk

If the radius $R$ of the charged disk we considered last time is very large compared to $z$, then the field along its axis winds up not depending upon $z$ at all:

$$E(z) = 2\pi k\sigma \hat{z} \left( 1 - \frac{z}{\sqrt{z^2 + R^2}} \right) \approx 2\pi k\sigma \hat{z}$$

-- that is, a uniform electric field.
How to make a uniform electric field

This last result suggests a nice way to make a uniform electric field, should you ever need one in the lab: two uniformly-charged parallel planes, with distance small compared to length or width. Seen edge on:

\[ E(z) = 4\pi k\sigma \hat{z} \]

That is, just superpose the fields from two oppositely-charged planes. They add in between the planes, and cancel outside.
The electric dipole

Appearing almost as frequently as point charges in physics homework are electric dipoles: equal and opposite point charges separated by a finite distance.

They appear in nature too: most molecules have overall charge separations, and thus dipole moments. Like water, depicted at right, for which

\[ p = 6 \times 10^{-30} \text{ coul m.} \]
Dipole in a uniform field

- If you put a free charge in a uniform electric field, it will accelerate, according to $F = ma$.
- Dipoles are neutral overall so they don’t accelerate. But the separation of charge leads to a torque, which we will now calculate.
Torque on a dipole in a uniform field

First the positive charge. The torque from its force about the center of the dipole, according to what you learned in PHY 121, is

\[\tau_+ = r_+ \times F_+\]

which, in terms of the angle \(\theta\) between the displacement and the force, is

\[|\tau_+| = r_+ F_+ \sin \theta\]

\[= \frac{\ell}{2} Q E \sin \theta = \frac{p}{2} E \sin \theta\]

\[E = E\hat{y}\]
Torque on a dipole in a uniform field (cont’d)

and its direction is into the page, according to the right-hand rule. In terms of the coordinate system we’ve indicated,

\[ \tau_+ = -\hat{z} \frac{p}{2} E \sin \theta \]

Next the negative charge. Since the magnitudes of \( r_- \) and \( F_- \) are the same as \( r_+ \) and \( F_+ \),

\[ |\tau_-| = |\tau_+| = \frac{p}{2} E \sin \theta \]
Torque on a dipole in a uniform field (cont’d)

By the right-hand rule, the direction of this torque is also the same as that of the previous one, so

$$\tau_\perp = -\hat{z} \frac{p}{2} E \sin \theta$$

Thus the total is

$$\tau = -\hat{z} p E \sin \theta$$

The torque is minimized by aligning the dipole moment $p$ with $E$ (i.e. $\theta = 0$).
Torque on a dipole in a uniform field (cont’d)

And this in turn can be written in a suggestive vector form:

\[ \tau = p \times E \]

Thus the utility of the dipole moment concept:

- fields exert torques on dipoles, as they exert forces on charges…
- and thus change the angular momentum of the dipole, as they change the linear momentum of charges.

\[ E = E\hat{y} \]