Today in Physics 122: V, and E from V

- Example electric potential calculations
- Potential as a shortcut in E calculations: examples
- Equipotential surfaces
- Potential and conductors
Example: field along the axis of a charged disk

Now for an illustration of getting $E$ by getting $V$ first and taking its derivative:

A circular disk with radius $R$ carries a charge $Q$ which is uniformly distributed on the disk. Calculate the electric field a distance $z$ above the center of the disk.

- It’s a 2-D charge distribution:

$$\sigma = \frac{Q}{\pi R^2}$$

and we can place the coordinate origin at the center of the disk with the $z$ axis pointing up.
Field along the axis of a charged disk (continued)

Since we’re only concerned with the axis of the disk, we can choose an infinitesimal charge element that’s symmetric about the axis:

\[ dQ = \sigma dA = 2\pi \sigma r' dr' \quad |r - r'| = \sqrt{r'^2 + z^2} \]

\[ V(z) = \int \frac{kdQ}{|r - r'|} = 2\pi k \sigma \int_0^R \frac{r' dr'}{\sqrt{r'^2 + z^2}} \]

Substitute:

\[ u = r'^2 + z^2 \quad \Rightarrow \quad du = 2r' dr' \]

\[ r' = 0 \rightarrow R \]

\[ \Rightarrow \quad u = z^2 \rightarrow z^2 + R^2 \]
Field along the axis of a charged disk (continued)

- After substituting, we have the potential:

\[ V(z) = \pi k\sigma \int_{z^2}^{z^2+R^2} \frac{du}{\sqrt{u}} = \pi k\sigma \left[ \frac{\sqrt{u}}{1/2} \right]_{z^2}^{z^2+R^2} \]

\[ = 2\pi k\sigma \left( \sqrt{z^2 + R^2} - z \right) \]

- And we get the field by differentiation:

\[ E(z) = -\hat{z} \frac{dV}{dz} = -2\pi k\sigma \hat{z} \left( \frac{1}{2} \frac{1}{\sqrt{z^2 + R^2}} \right) \left( 2z - 1 \right) \]

\[ = -2\pi k\sigma \hat{z} \left( \frac{1}{2} \frac{1}{\sqrt{z^2 + R^2}} \frac{z}{\sqrt{z^2 + R^2}} - 1 \right) = 2\pi k\sigma \hat{z} \left( 1 - \frac{z}{\sqrt{z^2 + R^2}} \right) \]

Compare to lecture on 6 September
Units of electric potential

The MKS unit of electric potential is the **volt**:

$$1 \text{ volt} = 1 \text{ joule coulomb}^{-1} = 1 \text{ kg m}^2 \text{ sec}^{-2} \text{ coul}^{-1}$$

The units of electric field are often expressed in voltage terms:

$$1 \text{ N coulomb}^{-1} = 1 \text{ joule m}^{-1} \text{ coulomb}^{-1} = 1 \text{ volt m}^{-1}$$

Since electrons and protons are often accelerated across regions with given voltage, a convenient energy unit also comes from the volt:

$$1 \text{ electron volt (eV)} = e \times 1 \text{ volt} = 1.6022 \times 10^{-19} \text{ joule}.$$
Recipe for $V$

As long as the charges do not extend to infinity:

- Determine the number of dimensions of the charge distribution.
- Choose a coordinate system that suits the geometry of the charge distribution and the location at which you need to know the field, and express the charge density in that coordinate system. (Usually this will still be a Cartesian coordinate system.)
- Express the infinitesimal element $dQ$, and the distance $|r - r'|$ from the infinitesimal element to the “test” point, in that coordinate system.
- Do the integral: $V = k \int dQ/|r - r'|$. 

Example: semicircular arc of charge

A semicircular wire with radius $R$ carries a total charge $Q$. Calculate the potential on the semicircle’s axis.

1-D charge distribution:

$$\lambda = \frac{Q}{\pi R},$$

which lies in a plane. Choose the $x$-$y$ plane for it to occupy, and place the origin at the semicircle’s center.

$$V(x,y,z) = ?$$
Semicircular arc of charge (continued)

- It’s easiest to envision the infinitesimal charge in polar coordinates:
  \[ R = \sqrt{x'^2 + y'^2}, \quad \phi = \arctan\left(\frac{y'}{x'}\right), \]
  so that \[ dQ = \lambda R d\phi, \]
  \[ \phi = 0 \rightarrow \pi, \]

- Distance between charge element and test point:
  \[ |r - r'| = \sqrt{R^2 + z^2} \]
Semicircular arc of charge (continued)

So the potential at $z$ is

\[
V = \int \frac{kdQ}{|r-r'|} = \frac{kR\lambda}{\sqrt{R^2+z^2}} \int_0^\pi d\phi
\]

\[
= \frac{\pi kR\lambda}{\sqrt{R^2+z^2}} = \frac{kQ}{\sqrt{R^2+z^2}}
\]

as you may have found intuitively obvious.
Example: along the axis of a cone

A conical surface carries a uniform charge density $\sigma$, its height is $h$, and $h$ is also the radius of its base. Find the potential difference between the cone’s vertex, $a$, and the center of its base, $b$.

- Divide it up into rings:
  - each ring has radius $s = r/\sqrt{2}$,
  - and an infinitesimal area bounded by this ring is
  \[
da = 2\pi s dr = \sqrt{2}\pi r dr
\]
  so
  \[
dQ = \sigma \sqrt{2}\pi r dr
\]
- The maximum distance along the cone’s surface from $a$ is
  \[
r = \sqrt{2}h.
\]
Along the axis of a cone (continued)

- So the potential at the vertex is \( V(a) = \sigma \sqrt{2\pi} \int_0^{\sqrt{2h}} dr = 2\pi \sigma h. \)
- Now for the base: its center lies a distance from \( dQ \) given by

\[
(r - r')^2 = \left( h - \sqrt{r^2 - s^2} \right)^2 + s^2
\]

\[
= \left( h - \frac{r}{\sqrt{2}} \right)^2 + \frac{r^2}{2}
\]

\[
= h^2 + \frac{r^2}{2} - \sqrt{2}hr + \frac{r^2}{2}
\]

\[
= h^2 + r^2 - \sqrt{2}hr
\]
Along the axis of a cone (continued)

so \( V(b) = \sqrt{2} \pi \sigma \int_{0}^{\sqrt{2}h} \frac{r \, dr}{\sqrt{h^2 + r^2 - \sqrt{2}hr}} \)

\[ = \sqrt{2} \pi \sigma h \int_{0}^{\sqrt{2}} \frac{u \, du}{\sqrt{u^2 - \sqrt{2}u + 1}} \]

Now for a useful trick. Complete the square in the denominator:

\[ u^2 - \sqrt{2}u + 1 = u^2 - \sqrt{2}u + \frac{1}{2} + \frac{1}{2} \]

\[ = \left(u - \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2} \]

Substitute \( u = \frac{r}{h} \), \( du = \frac{dr}{h} \):

\[ \int \frac{r \, dr}{\sqrt{h^2 + r^2 - \sqrt{2}hr}} \]

\[ = \sqrt{2} \pi \sigma h \int \frac{u \, du}{\sqrt{u^2 - \sqrt{2}u + 1}} \]

\[ = \left(u - \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2} \]
Along the axis of a cone (continued)

Now substitute $v = u - \frac{1}{\sqrt{2}}$, $dv = du$, $v = -\frac{1}{\sqrt{2}} \rightarrow \frac{1}{\sqrt{2}}$:

$$V(b) = \sqrt{2}\pi \sigma h \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{(v + 1/\sqrt{2}) dv}{\sqrt{v^2 + 1/2}}.$$

First part: $w = v^2 + 1/2$, $dw = 2vdv$, whence

$$\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{vdv}{\sqrt{v^2 + 1/2}} = \frac{1}{2} \int_{1}^{1} \frac{dw}{\sqrt{w}} = 0.$$

Second part: $v = (\tan \theta)/\sqrt{2}$, $dv = \left(\sec^2 \theta \, d\theta\right)/\sqrt{2}$
Along the axis of a cone (continued)

\[
\frac{1}{\sqrt{2}} \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{dv}{\sqrt{v^2 + 1/2}} = \frac{1}{2} \int_{-\pi/4}^{\pi/4} \frac{\sec^2 \theta \, d\theta}{\sqrt{\tan^2 \theta + 1/2}}
\]

\[
= \frac{1}{\sqrt{2}} \int_{-\pi/4}^{\pi/4} \frac{\sec^2 \theta \, d\theta}{\sqrt{\tan^2 \theta + 1}} = \frac{1}{\sqrt{2}} \int_{-\pi/4}^{\pi/4} \sec \theta \, d\theta
\]

The last integral requires a really dirty trick, so I’ll just give the answer:

\[
\frac{1}{\sqrt{2}} \int_{-\pi/4}^{\pi/4} \sec \theta \, d\theta = \sqrt{2} \ln\left(\sqrt{2} + 1\right)
\]
Along the axis of a cone (continued)

Thus

\[ V(b) = \sqrt{2\pi}\sigma h \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{(v + 1/\sqrt{2}) \, dv}{\sqrt{v^2 + 1/2}} = 2\pi\sigma h \ln\left(\sqrt{2} + 1\right) , \]

and

\[ V(a) - V(b) = 2\pi\sigma h \left[ 1 - \ln\left(\sqrt{2} + 1\right) \right] \]

(We won’t give you anything this hard on the test.)
Recipe for $E$ from $V$

If you know $V$ and want $E$:

- Calculate the derivatives of $V$ with respect to each coordinate, to get the component of $E$ corresponding to that coordinate:

\[
E = -\nabla V
\]

\[
E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z}
\]

That's it.

- **Caveat:** one will need to know $V$ all over three-dimensional space in order to calculate the derivatives properly.
Example: potential from a dipole

Most potential problems involving point charges are quite straightforward. Here, for example, the dipole:

\[
V = kQ \left( \frac{1}{\sqrt{(z - \ell/2)^2 + x^2}} - \frac{1}{\sqrt{(z + \ell/2)^2 + x^2}} \right)
\]

And \( E \) is hardly ever difficult to calculate, although it may take a lot of writing:
Example: field from the potential from a dipole

\[ E_x = -kQ \frac{\partial}{\partial x} \left( \frac{1}{\sqrt{(z - \ell/2)^2 + x^2}} - \frac{1}{\sqrt{(z + \ell/2)^2 + x^2}} \right) \]

\[ = -kQ \left( -\frac{1}{2} \frac{1}{\left[ (z - \ell/2)^2 + x^2 \right]^{3/2}} 2x + \frac{1}{2} \frac{1}{\left[ (z + \ell/2)^2 + x^2 \right]^{3/2}} 2x \right) \]

\[ = kQx \left( \frac{1}{\left[ (z - \ell/2)^2 + x^2 \right]^{3/2}} - \frac{1}{\left[ (z + \ell/2)^2 + x^2 \right]^{3/2}} \right) \]
Field from the potential from a dipole (continued)

\[ E_z = -kQ \frac{\partial}{\partial z} \left( \frac{1}{\sqrt{(z-\ell/2)^2 + x^2}} - \frac{1}{\sqrt{(z+\ell/2)^2 + x^2}} \right) \]

\[ = -kQ \left( -\frac{1}{2} \frac{1}{\left[ (z-\ell/2)^2 + x^2 \right]^{3/2}} \frac{2(z-\ell/2) + \frac{1}{2} \frac{1}{\left[ (z+\ell/2)^2 + x^2 \right]^{3/2}} 2(z+\ell/2) \right) \]

\[ = -kQ \left( \frac{z + \ell/2}{\left[ (z+\ell/2)^2 + x^2 \right]^{3/2}} - \frac{z - \ell/2}{\left[ (z-\ell/2)^2 + x^2 \right]^{3/2}} \right) \]
Field from the potential from a dipole (continued)

If we are far from the dipole (i.e. if \( z \) or \( x \gg \ell \)), and \( x = 0 \),

\[
E_z = -kQ \left( \frac{z + \ell/2}{(z + \ell/2)^{3/2}} - \frac{z - \ell/2}{(z - \ell/2)^{3/2}} \right)
\]

\[
= -\frac{kQ}{z^3} \left( \frac{z + \ell/2}{(1 + \ell/2z)^3} - \frac{z - \ell/2}{(1 - \ell/2z)^3} \right) \quad \text{Use } (1 + x)^n \approx 1 + nx, |x| \ll 1:
\]

\[
\approx -\frac{kQ}{z^3} \left[ (z + \ell/2)(1 - 3\ell/2z) - (z - \ell/2)(1 + 3\ell/2z) \right]
\]

\[
= -\frac{kQ}{z^3} \left[ z + \ell/2 - 3\ell/2 - 3\ell^2/2z - (z - \ell/2 + 3\ell/2 - 3\ell^2/2z) \right]
\]

\[
= -\frac{kQ}{z^3} \left[ -2\ell \right] = \frac{2kp}{z^3} \quad \text{Compare to Problem 21-67 in the textbook.}
\]
Equipotentials

Curves or surfaces with uniform value of potential are called **equipotentials**.

- Equipotentials never intersect.
- Their density is larger, the larger the value of $E$.
- Lines of $E$ are perpendicular to equipotentials ($E = -\nabla V$).
- **The surfaces of perfect conductors are equipotentials.**

- Since $E = 0$ everywhere within a conductor, and we are free to choose *any* path along which to integrate,

$$\Delta V = -\int_{a}^{b} E \cdot d\ell = 0$$

between any two points $a$ and $b$ on such a surface.
Equipotentials (continued)

Equipotentials and field lines for a dipole.

To generate your own, go to http://www.cco.caltech.edu/~phys1/java/phys1/EField/EField.html