Today in Physics 122: magnetic forces and fields

- Point charges, magnetic fields, and cyclotron motion
- Tension in, and torque on, current loops in uniform magnetic fields
- Magnetic fields from currents: the Biot-Savart field law
Brief reminders from the recent past

Force laws:
\[ dF = Id\ell \times B = Id\ell B \sin \theta \hat{n} \]
\[ F = \int Id\ell \times B = \int IB \sin \theta \hat{n} d\ell \]
\[ F = Qv \times B \quad \text{point charges} \]

Cross products:
\[ a \times b = -b \times a = ab \sin \theta \hat{n} \]
\[ \therefore a \times b = 0 \text{ if } a \parallel b \]
\[ \therefore a \times b = ab \hat{n} \text{ if } a \perp b \]
Cross products of Cartesian unit vectors:
\[ \hat{x} \times \hat{y} = \hat{z} \quad \hat{y} \times \hat{z} = \hat{x} \quad \hat{z} \times \hat{x} = \hat{y} \]

Id\ell \text{ or } Qv
\[ dF \text{ or } F \]
B
\[ \theta \]

Lay fingers of right hand along arc from \( d\ell \) to \( B \), pointing to \( B \). Thumb points along \( dF \).
A charge $Q$ with mass $m$ enters a region with uniform $B = -B \hat{z}$. We first notice it when it has velocity $\mathbf{v} = v \hat{x}$. How does it move subsequently?

- A magnetic force is exerted:
  \[ F = Qv \times B = QvB \hat{y} \]
  \[ a = F/m = QvB \hat{y}/m \]

- The sideways acceleration $a$ changes the direction of $\mathbf{v}$ but not its magnitude.
Force on a charge moving within uniform $B$

- Since the magnitude of $v$ never changes, neither does the magnitude of $a$. The charge continues to turn left at a constant rate.
- You have seen these conditions before, in PHY 121. They are those for uniform circular motion:

$$F = ma = -\frac{mv^2}{r} \hat{r}.$$
So the charge moves in a circle, with radius given by

\[ F = Qv \times B = -QvB \hat{r} \]

\[ = ma = - \frac{mv^2}{r} \hat{r} \]

\[ r = \frac{mv}{QB} \quad \text{Cyclotron radius} \]

This concept has many important implications, of which we may mention two.
1. Cyclotron motion is the basis of mass spectroscopy, which is our means of measuring atomic abundances very precisely.

- Electron/ion gun, focused on sample
- Uniform $B$
- Beam of ions sputtered from sample
- Single-ion detectors
- $Q/m$: small, large
- Electronic counters
- Mineral sample

<table>
<thead>
<tr>
<th>86Sr</th>
<th>87Sr</th>
<th>87Rb</th>
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</table>
| 1 0 5 8 0 3 4 | 0 7 4 1 0 1 5 | 0 0 7 9 8 8 2
Force on a charge moving within uniform $B$ (continued)

2. As you will discover in PHY 123, accelerating electric charges radiate light. Thus charges undergoing cyclotron motion emit **cyclotron radiation**. Applications:

a. Free electrons in motion, and significant magnetic fields, are present throughout interstellar space; thereby, cyclotron radiation dominates the emission by galaxies at radio frequencies. (Ask Dan.)

VLA image of the radio galaxy Cygnus A ([Rick Perley, NRAO]).
b. Cyclotron radiation is larger, the larger is the magnetic acceleration; which in turn is larger if $Q/m$ is larger. Thus ring-shaped particle accelerators like the Large Hadron Collider (LHC), which steer the particles in circles using $B$, accelerate heavy particles like protons. To use electrons, while avoiding prohibitive cyclotron radiation, elementary-particle physicists build linear colliders like SLAC. (Ask Regina.)
Force on a current immersed in $B$

Earth, and its immediate vicinity, are bathed in a magnetic field that is uniform on scales up to mountain-size, so this situation is easily recreated:

- A square loop of wire, with side $L$, is immersed in a uniform magnetic field with magnitude $B$ and direction perpendicular to the loop, as shown. What is the net force on the loop?

$B$ (into page)
Based on our experience last time you can probably guess the answer, but we’ll go through it anyway to remind ourselves of the recipe.

- Use a coordinate system aligned with the loop; then $dl$ can be taken as $dx$ or $dy$ for horizontal or vertical segments.

- $B$ points along $-z$ as shown, so $\sin \theta = 1$ in the cross products (i.e. $Idl \perp B$).
Force on a current immersed in $B$ (continued)

 Forces on the segments:

\[ F_1 = \int I dl \times B = \int I dy \hat{y} \times (-B \hat{z}) \]
\[ = -IB(\hat{y} \times \hat{z}) \int dy = -IBL\hat{x} \]

\[ F_2 = \int I dx \hat{x} \times (-B \hat{z}) \]
\[ = IBL(\hat{z} \times \hat{x}) = IBL\hat{y} \]
Force on a current immersed in $B$ (continued)

... and

\[ F_3 = \int_0^L I dy (-\hat{y}) \times (-B\hat{z}) \]
\[ = IB(\hat{y} \times \hat{z}) \int_0^L dy = IBL\hat{x} \]

\[ F_4 = \int_0^L I dx (-\hat{x}) \times (-B\hat{z}) \]
\[ = IBL(\hat{x} \times \hat{z}) = -IBL\hat{y} \]
So, as you no doubt expected,

\[ F = F_1 + F_2 + F_3 + F_4 = -IBL\hat{x} + IBL\hat{y} + IBL\hat{x} - IBL\hat{y} = 0 \]

Is there a loop shape, besides square, that would give a nonzero \( F \) in a uniform \( B \)?

- Yes: a circle would.
- No.
No, a circle would not.

Circular loop:

- Appropriate $Id\ell = IRd\phi$; $\phi$ ranges from 0 to $2\pi$.
- $Id\ell$ still $\perp B$; $\sin \theta = 1$, and $\hat{n} = \cos \phi \hat{x} + \sin \phi \hat{y}$.
- Thus

$$F = \int Id\ell \times B = \int IRd\phi B \sin \theta \hat{n}$$

$$= IBR \int_{0}^{2\pi} (\hat{x} \cos \phi + \hat{y} \sin \phi) \, d\phi$$

$$= IBR \left[ \hat{x} \sin \phi - \hat{y} \cos \phi \right]_{0}^{2\pi} = 0$$
Tension in a current loop immersed in $B$

Back to the square loop: $F = 0$ means none of the segments move. Consider the equilibrium of one segment:

- From that segment’s viewpoint, the reason it doesn’t move is that the magnetic force is balanced by the tension in the loop. (Just like a mass hanging from a rope, for which the force of gravity is balanced by rope tension.)
Tension in a current loop immersed in $B$ (continued)

- The tension is thus pretty easy to work out:
  \[ F_2 + 2T = IBL\hat{y} + 2T = 0 \]
  \[ T = \frac{IBL}{2} \]

- Similarly for a circle:
  balance the magnetic force on half of it – a semicircle, treated above and in Tuesday’s lecture – with tension from the other half:
  \[ F_{s-c} + 2T = 2IBR\hat{y} + 2T = 0 \]
  \[ T = \frac{IBR}{2} \]
Torque on a current loop immersed in $B$

Now let’s pivot the loop about the $x$ axis, by an angle $\theta$. (And twist our view too.)

- This changes the angle between the current and the field in the vertical segments, thus changing $F_1$ and $F_3$. But these forces still lie in the plane of the loop:

\[
F_1 = -IBL \sin(90^\circ + \theta) \hat{x} = -IBL \cos \theta \hat{x}
\]
\[
F_3 = IBL \sin(90^\circ - \theta) \hat{x} = IBL \cos \theta \hat{x} = -F_1
\]
Torque on a current loop immersed in $B$ (continued)

- The pivot changes neither the angle between horizontal currents and field, nor $F_2$ and $F_4$. But now $F_2$ and $F_4$ do not lie in the plane of the loop: they produce a torque.

$$\tau = r_2 \times F_2 + r_4 \times F_4$$

$$= -\frac{L}{2} IBL \sin \theta \hat{x}$$

$$- \frac{L}{2} IBL \sin \theta \hat{x}$$

$$= -IL^2B \sin \theta \hat{x}$$
Torque on a current loop immersed in $B$
(continued)

In analogy with the electric dipole moment, we can construct a magnetic dipole moment for the loop:

$$A = L^2 \hat{n} = L^2 (\hat{z} \cos \theta + \hat{y} \sin \theta)$$

$$\mu = IA = IL^2 (\hat{z} \cos \theta + \hat{y} \sin \theta)$$

$$\tau = \mu \times B = IL^2 (\hat{z} \cos \theta + \hat{y} \sin \theta) \times (-B \hat{z})$$

$$= -IL^2 B \sin \theta (\hat{y} \times \hat{z})$$

$$= -IL^2 B \sin \theta \hat{x}$$

Fingers of RH along $I \Rightarrow$ thumb along $A$

$\theta$

$\tau$ (out of page)

$B$

$z$

$y$

$A$

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So a current loop which carries current $I$ and has area $A$ has a magnetic dipole moment $\mu = IA$, on which an external magnetic field $B$ exerts a torque,

$$\tau = \mu \times B,$$

just as an external electric field $E$ exerts a torque on an electric dipole moment $p$:

$$\tau = p \times E.$$
Fields from currents

Biot and Savart followed up their discovery of magnetic forces on currents by characterizing the generation of fields by currents, discovered by Ørsted. What they found is now called the Biot-Savart field law:

\[
dB = \frac{\mu_0}{4\pi} \frac{Id\ell \times (r - r')}{|r - r'|^2}
\]

where \( \mu_0 = 4\pi \times 10^{-7} \text{ T m A}^{-1} \)

Compare to Coulomb’s Law:

\[
dE = \frac{1}{4\pi\varepsilon_0} \frac{dQ(r - r')}{|r - r'|^2}
\]
Fields from currents (continued)

The theory reproduces Ørsted’s results, for the field from a long straight wire (infinite; only a small bit shown at right) carrying current $I$:

$$d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{I \mathbf{d}\ell \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^2} = \frac{\mu_0}{4\pi} \frac{I \mathbf{d}z \sin \theta}{z^2 + R^2} \hat{x}$$

Eliminate $z$ or $\theta$. I’ll keep $\theta$.

$$\sin \theta = \frac{R}{\sqrt{z^2 + R^2}}$$

$$\cos \theta d\theta = -\frac{1}{2} \frac{R}{\left(\frac{z^2}{Z^2} + R^2\right)^{3/2}} 2z dz$$

$$= \frac{1}{R} \sin^2 \theta \cos \theta dz$$

$$I \mathbf{d}\ell = I \mathbf{d}z \hat{z}$$
Fields from currents (continued)

The angle $\theta$ runs from zero ($z = -\infty$) to $\pi$ ($z = +\infty$). Thus

$$B(r) = \hat{x} \frac{\mu_0 I}{4\pi} \int_0^\pi \left( \frac{\sin^3 \theta}{R^2} \right) \left( \frac{R d\theta}{\sin^2 \theta} \right)$$

$$= -\hat{x} \frac{\mu_0 I}{4\pi} \frac{1}{R} \cos \theta \left|_0^\pi \right.$$ 

$$= -\hat{x} \frac{\mu_0 I}{4\pi} \frac{1}{R} (-1 - 1)$$

$$= \hat{x} \frac{\mu_0 I}{2\pi R}$$

which is what Ørsted had found.
Since the current distribution is cylindrically symmetric, we would have the same answer no matter which direction we pointed the y axis: $B$ would point out of the plane containing the wire and $r$, making circles around $I$. So we can in fact write

$$B(r) = \frac{\mu_0 I}{2\pi R} \hat{\phi},$$

where $\hat{\phi}$ is a unit vector pointing in the direction that the polar coordinate $\phi$ increases.