Today in Physics 122: the Maxwell equations

- Ampère’s law is broken, and Maxwell fixes it.
- Gauss’s law for magnetic fields.
- The Maxwell equations.
- The Maxwell equations in vacuum lead to a wave equation.

Pretty electromagnetic waves: a rainbow over the Potala Palace, Lhasa, Tibet; photo by Galen Rowell.
Displacement current

When we introduced Ampère’s law we carefully noted that it applies in the given form only for constant currents.

And then we went and applied it to AC currents but were even more careful in our choice of situations.

Here’s why. Consider a long straight wire, an AC current, and a parallel-plate capacitor (plate separation exaggerated here for clarity).
Displacement current (continued)

To calculate $B$ from the current, at a place far from the capacitor, we can draw a circular Ampèrean loop $C$ as shown.

- Usually we take $C$ to enclose a coplanar circular area $S_1$.
- This area encloses the current $I$, and Ampère’s law becomes

\[
\oint_{C} \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_{S_1} \mathbf{J} \cdot d\mathbf{A} = \mu_0 I_{\text{encl}} = \mu_0 I,
\]
Displacement current (continued)

whence
\[ \oint_{C} \mathbf{B} \cdot d\ell = B2\pi r = \mu_0 I \quad \Rightarrow \quad \mathbf{B} = \frac{\mu_0 I}{2\pi r} \hat{\phi} \]

But \( S_1 \) isn’t the only surface bounded by \( C \). Consider \( S_2 \), which passes cleanly between the plates of the capacitor, and for which \( I_{\text{encl}} = \int_{S_2} \mathbf{J} \cdot d\mathbf{A} = 0 \).
Displacement current (continued)

- So using $S_2$, $B = 0$ on $C$, while it's not zero on $C$ using $S_1$.

A way out of the inconsistency: note that the electric field is continuous across the capacitor, and the form this takes is reminiscent of a current:

$$I = I_0 e^{i\omega t} \quad \Rightarrow \quad Q = \frac{I_0}{i\omega} e^{i\omega t}$$

$$E = 4\pi k\sigma = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A} = \frac{I_0}{i\omega \varepsilon_0 A} e^{i\omega t}$$

$$\Phi_E = \frac{I_0}{i\omega \varepsilon_0} e^{i\omega t}$$

$$\varepsilon_0 \frac{d\Phi_E}{dt} = I_0 e^{i\omega t}$$
That is, $\varepsilon_0 \frac{d\Phi_E}{dt}$ preserves the trace of the current through the vacuum of the capacitor’s gap, and is absent elsewhere in the circuit.

☐ If this term is added to the current, then $B$ on $C$ comes out the same independent of which surface is used. And thus a repaired form of Ampère’s law with that inconsistency removed:

$$\oint_C \mathbf{B} \cdot d\ell = \mu_0 I_{\text{encl}} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} = \mu_0 \int_S \left( \mathbf{J} + \varepsilon_0 \frac{\partial E}{\partial t} \right) \cdot dA$$

Displacement current
Displacement current (continued)

This repair to Ampère’s law was made by James Clerk Maxwell, who also named the extra term “displacement current.”

- Maxwell’s motivation was different from ours: he invented displacement current not to remove the “capacitor” inconsistency in $B$ calculations, but because there was otherwise no way for the equations of electricity and magnetism to lead to a wave equation for the propagation of electromagnetic energy (see below).
- The real proof that displacement current is required, though, was also realized by Maxwell. If you can’t wait til your junior-level fields class to learn why, look [here](#).
Implications of displacement current: changes in $E$ can induce $B$

Faraday’s Law was evidence that EMFs and electric fields can be induced by time-variable magnetic fields. Displacement current indicates that the converse is also true.

- Consider the **interior** of the capacitor (c.f. example 31.1 in Giancoli):

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} = \mu_0 \varepsilon_0 \frac{d}{dt} \int_S \mathbf{E} \cdot d\mathbf{A}$$

$$B2\pi r = \mu_0 \varepsilon_0 \frac{d}{dt} \left( \frac{I_0}{i\omega\varepsilon_0 A} e^{i\omega t} \right) \pi r^2 = \mu_0 \frac{I_0}{A} e^{i\omega t} \pi r^2$$

$$\mathbf{B} = \frac{\mu_0 I_0 r}{2A} e^{i\omega t} \hat{\phi} \quad \text{B induced in free space by E.}$$
Equations of electromagnetism so far

With the repair, the integral relations among the fields, charges and currents we have so far are Gauss’s, Faraday’s and Ampère’s laws:

\[ \oint E \cdot dA = \frac{Q_{\text{encl}}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int \rho \, dv \]

\[ \oint E \cdot d\ell = - \frac{d\Phi_B}{dt} = - \frac{d}{dt} \int_S \mathbf{B} \cdot dA \]

\[ \oint B \cdot d\ell = \mu_0 I_{\text{encl}} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} = \mu_0 \int_S \left( \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \cdot dA \]
Equations of electromagnetism so far (continued)

There’s almost a symmetry:

\[ \oint_S E \cdot dA = \frac{Q_{encl}}{\varepsilon_0} \]

\[ \oint_C E \cdot d\ell = -\frac{d\Phi_B}{dt} \]

\[ \oint_C B \cdot d\ell = \mu_0 I_{encl} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} \]

All that’s missing is a Gauss’s Law for magnetism: a complete integral of magnetic flux over a closed surface. For this, recall the following results about the flux of \( E \) through closed surfaces, which we covered on 13 September 2012:
Nonuniform field, irregular surface

Even if the field varies in strength with position, and the surface is irregular, one can always go to the location of each infinitesimal area element in the surface and

- find the local value of $E$
- define an area vector $dA$ for the area element.

Then the total flux through that surface is the sum of the fluxes through all the infinitesimal elements:

$$\Phi = \int_{\text{surface}} E \cdot dA$$

Don’t worry, we’ll only be using simple surfaces that lead to do-able integrals.
Closed surfaces and fields from charges

Use your intuition. Electric field lines from charges originate at + charges and terminate at – charges. So which of these spheres has a non-zero flux through it?

1) Left 2) Right 3) Both
Closed surfaces and fields from charges (continued)

This sphere contains an electric dipole, but the + (red) charge is closest to its inner surface. The flux through it is

1) positive

2) negative

3) zero
Closed surfaces and fields from charges (continued)

Evidently, the flux of $E$ through a closed surface depends upon how much charge it contains, since lines of $E$ can only start and finish on charges.

- For such a flux to be positive (negative), it needs to contain an net positive (negative) charge.
- If it contains no net charge, every flux line that enters the surface has to leave it too.
Equations of electromagnetism so far (continued)

The point, of course, is that there is no such thing as magnetic charge.

- The lowest order of magnetic entity is the dipole, and for electric dipoles, the flux is zero through any surface containing the dipole.

Thus the magnetic flux through any closed surface is zero:

\[ \oint_S \mathbf{B} \cdot d\mathbf{A} = 0 \]

Gauss’s Law for magnetism

And this leads to a pleasing symmetry among the equations of electromagnetism.
The Maxwell equations

\[ \oint_E \cdot dA = \frac{Q_{\text{encl}}}{\varepsilon_0} \quad \oint_B \cdot dA = 0 \]

\[ \oint_E \cdot dl = -\frac{d\Phi_B}{dt} \quad \oint_B \cdot dl = \mu_0 I_{\text{encl}} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} \]

This is the integral form of the master set of equations which describe electromagnetism.

- Note that all were discovered separately, and though by many to pertain to different effects.
- The discovery of this form of the four equations by Maxwell was the final demonstration that electricity and magnetism are two aspects of the same physical effect.
A math interlude

Not everybody in this class has had MTH 164 yet, but many have. Since we won’t use the process of what I’m about to do again in PHY 122, it won’t cause harm to borrow two facts from MTH 164. Those who have not had, or are not currently taking, MTH 164 are encouraged to doze until our final result.

The integral forms of the Maxwell equations turn out not to be terribly useful for solving complex problems. But with a couple of integral theorems involving vector calculus:

- Gauss’s divergence theorem
- Stokes’s curl theorem

we can rearrange them into a differential form that is.
A math interlude (continued)

Recall the gradient operator in 3-D:

\[ \nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \]

For any vector function \( f \),

\[ \oint_{S} f \cdot dA = \int_{\mathcal{V}} \nabla \cdot f \, dv \quad \text{Gauss's divergence theorem} \]

\[ \oint_{C} f \cdot dl = \int_{S} (\nabla \times f) \cdot dA \quad \text{Stokes's curl theorem} \]

where closed surface \( S \) bounds volume \( \mathcal{V} \), and closed curve \( C \) bounds surface \( S \).
A math interlude (continued)

With these, the first two Maxwell equations become

\[ \oint E \cdot dA = \int \nabla \cdot E \, dv = \frac{Q_{\text{encl}}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int \rho \, dv, \]

or \[ \nabla \cdot E = \frac{\rho}{\varepsilon_0}. \]

\[ \oint B \cdot dA = \int \nabla \cdot B \, dv = 0, \]

or \[ \nabla \cdot B = 0. \]

where, again, \( \rho \) is the electric charge density at the point in space at which the derivatives of the fields are taken.
A math interlude (continued)

For the second pair,
\[ \oint E \cdot d\ell = \int (\nabla \times E) \cdot dA = -\frac{d\Phi_B}{dt} = -\frac{d}{dt} \int B \cdot dA, \]

or \[ \nabla \times E = -\frac{\partial B}{\partial t}. \]

\[ \oint B \cdot d\ell = \int (\nabla \times B) \cdot dA = \mu_0 I_{\text{encl}} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} = \mu_0 \int_S \left( J + \varepsilon_0 \frac{\partial E}{\partial t} \right) \cdot dA, \]

or \[ \nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}. \]

Here \( J \) is the current density at the point in space at which the derivatives of the fields are taken.
A math interlude (continued)

Thus, in differential form, the Maxwell equations are

\[ \nabla \cdot E = \frac{\rho}{\varepsilon_0} \]

\[ \nabla \cdot B = 0 \]

\[ \nabla \times E = -\frac{\partial B}{\partial t} \]

\[ \nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} \]

Let's suppose we're in vacuum, with no charges or currents around. The Maxwell equations reduce to

\[ \nabla \cdot E = 0 \]

\[ \nabla \cdot B = 0 \]

\[ \nabla \times E = -\frac{\partial B}{\partial t} \]

\[ \nabla \times B = \frac{1}{c^2} \frac{\partial E}{\partial t} \]

where \( \frac{1}{c^2} = \varepsilon_0 \mu_0 \).
A math interlude (continued)

Now take the curl of Faraday’s law:

$$\nabla \times (\nabla \times E) = -\frac{\partial (\nabla \times B)}{\partial t} = -\frac{1}{c^2} \frac{\partial^2 E}{\partial t^2}$$

subbing in Ampère’s law

and use the triple-vector-product (BAC-CAB) identity on the left-hand side:

$$\nabla \times (\nabla \times E) = \nabla (\nabla \cdot E) - \nabla^2 E = -\frac{1}{c^2} \frac{\partial^2 E}{\partial t^2}$$

0, by Gauss’s law

$$\nabla^2 E = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) E = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2}$$

Wave equation for $E$
A math interlude (continued)

For simplicity let’s reduce this to one dimension of length, say $z$:

$$\frac{\partial^2 E}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2}$$

This sort of differential equation, which relates second-order space derivatives to second-order time derivatives, is called a wave equation. To solve it is in the domain of MTH 165, MTH 281 or ME 201, but since this an interlude:

- Suppose that we focus on one component of $E$. Suppose further that this component is given by

$$E(z, t) = Z(z) T(t)$$
A math interlude (continued)

Then, since derivatives w.r.t. $z$ don’t operate on $T$, and those w.r.t. $t$ don’t operate on $Z$,

$$T \frac{d^2 Z}{dz^2} = \frac{Z}{c^2} \frac{d^2 T}{dt^2}$$

Divide through by $ZT$:

$$\frac{1}{Z} \frac{d^2 Z}{dz^2} = \frac{1}{c^2} \frac{1}{T} \frac{d^2 T}{dt^2} = \text{constant} = -\kappa^2$$

since two functions of different variables can’t be equal for any choice of those variables, unless those functions are constants.
A math interlude (continued)

Thus we are split into two equations:

\[ \frac{d^2 Z}{dz^2} + \kappa^2 Z = 0 \quad \frac{d^2 T}{dt^2} + \kappa^2 c^2 T = 0 \]

which, finally, look familiar: they are both like the equation of simple harmonic motion. The solutions include

\[ Z(z) = Ae^{i\kappa z} \quad T(t) = Be^{i\kappa c t} \equiv Be^{i\omega t} \]

So, bundling up all of the leading factors, the electric field which solves the wave equation is

\[ E(z, t) = Z(z)T(t) = E_0 e^{i(\kappa z + \omega t)} \]  

**Electromagnetic wave**