Today in Physics 122: forces and signals from electromagnetic waves

- Momentum in electromagnetic plane waves
- Radiation pressure
- Wave modulation and group velocity

Artist’s conception of a “solar sail:” a spacecraft propelled by solar radiation pressure. (Benjamin Diedrich, Caltech.)
Recap: electromagnetic plane waves

In vacuum with no charges or currents present, electric and magnetic fields can exist as travelling waves, which have $E \perp B$ and in phase (peaks line up), which are transverse, and which travel in the direction of $E \times B$:

$$E(z,t) = \hat{x}E_0 e^{i(kz \pm \omega t)}$$
$$B(z,t) = \mp \hat{y} \frac{E_0}{c} e^{i(kz \pm \omega t)}$$

Waves carry energy per unit time and area given by the Poynting vector,

$$S = \frac{1}{\mu_0} E \times B = \hat{z}cu$$

where $u = \frac{1}{2} \left( \varepsilon_0 E^2 + B^2 / \mu_0 \right)$ is the energy density in the fields.
Recap: electromagnetic plane waves (continued)

Relations among quantities that determine the wave’s pattern:

\[ \frac{\omega}{\kappa} = f \lambda = c \]
\[ \lambda = \frac{2\pi}{\kappa} \]
\[ \tau = \frac{2\pi}{\omega} = \frac{1}{f} \]

Phase velocity of wave (speed of light in vacuum):

\[ v = \frac{d \phi}{dt} = \frac{d}{dt} (\kappa z + \omega t) = 0 \]
\[ v = \frac{dz}{dt} = -\frac{\omega}{\kappa} = -c = -\frac{1}{\sqrt{\varepsilon_0 \mu_0}} \]
\[ c = 2.99792458 \times 10^8 \text{ m sec}^{-1} \]
Momentum carried by electromagnetic waves

Since electromagnetic waves carry energy it’s not surprising that they also carry momentum.

There are three ways one sees in use, for calculating the amount of momentum in such waves:

1. Borrow a couple of simple facts from PHY 123.
   - Electromagnetic waves can be thought of as streams of photons, the “particle” expressions of light: this is one consequence of the quantum-mechanical wave-particle duality.
   - Photons have energy $\varepsilon$ and momentum $p$ given by
     \[ \varepsilon = \frac{hc}{\lambda}, \quad p = \frac{h}{\lambda} = \frac{\varepsilon}{c} \]
So we can re-write the energy density in electric and magnetic fields as \( u = n\varepsilon \), where \( n \) is the number of photons per unit volume, and the Poynting vector becomes

\[
S = \frac{E \times B}{\mu_0} = \hat{z}cu = \hat{z}cn\varepsilon = \hat{z}c(n\rho)
\]

Just as \( \hat{z}cu \) is the energy per unit time and area which passes through a given \( x-y \) plane, \( \hat{z}cn\rho \) is the momentum per unit time and area which passes through that plane. So the vector which describes momentum transfer is just a factor of \( c \) different from the Poynting vector:

\[
\mathcal{P} = \frac{S}{c} = \sqrt{\frac{\varepsilon_0}{\mu_0}} E \times B = \hat{z}cn\rho
\]

Momentum per unit time and area, carried by electromagnetic waves
Momentum (continued)

This is a bit unsatisfying because it makes it look as if the momentum content of electromagnetic waves is a quantum-mechanical effect. It is not.

Alternatively, one can

2. return to the Lorentz force law and the Maxwell equations and calculate the momentum flow rate from scratch.

☐ This is the way Maxwell first got the momentum of EM waves. That his way involves the Maxwell stress tensor is a good illustration of how far beyond the scope of PHY 122 this method is.

So we will employ a halfway measure (see Crawford, Waves [Berkeley physics course, v. 3])…
Momentum (continued)

3. Time averaging

Consider the real parts of a wave propagating toward $+z$:

$$E(z, t) = \hat{x} E_0 \cos(\kappa z - \omega t) = E\hat{x}$$

$$B(z, t) = \hat{y} \frac{E_0}{c} \cos(\kappa z - \omega t) = B\hat{y}$$

and the force this exerts on a charge $Q$ initially at rest at time $t$:

$$F = QE + Qv \times B = QE\hat{x} + Q\left(v_x \hat{x} + v_y \hat{y} + v_z \hat{z}\right) \times B\hat{y}$$

$$= \left(QE - Qv_z B\right) \hat{x} + \left(Qv_x B\right) \hat{z}$$

Now observe:
The electric force is much larger than the magnetic force: 
\[ QvB = QEv/c, \] and \( v \ll c, \) at least at first. So 
\[ F_x(z,t) \gg F_z(z,t). \]

\( v_z \) increases at the rate \( a_z = F_z/m. \)

- So whatever \( v_z \) is, it is a speed much smaller than \( c \) at first, and grows slowly over the time scale of one period of the wave’s oscillation.

So let’s **average** the force over one period of oscillation of the fields:

\[
\langle F \rangle = \frac{1}{\tau} \int_0^\tau F(z,t)\,dt = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} F(z,t)\,dt
\]

In this process the slowly-changing \( v_z \) can be regarded as a constant.
Momentum (continued)

Note that
\[
\frac{\omega}{2\pi} \int_0^{2\pi/\omega} \cos(\kappa z - \omega t) \, dt = \frac{\omega}{2\pi} \left( \frac{1}{-i\omega} \sin[\kappa z - \omega t] \right)\bigg|_0^{2\pi/\omega} = \sin \kappa z
\]

\[= i \frac{1}{2\pi} \left( \sin[\kappa z - 2\pi] - \sin \kappa z \right) = 0\]

So:
\[
\langle F_x \rangle = Q \langle E \rangle - Q v_z \langle B \rangle
\]
\[
= (Q E_0 - Q E_0 v_z / c) \frac{2\pi/\omega}{2\pi} \int_0^{2\pi/\omega} \cos(\kappa z - \omega t) \, dt = 0
\]
Momentum (continued)

- But the other term does not vanish upon averaging, because it involves products of sines and cosines. So

\[
\langle F \rangle = \left\langle \frac{dP}{dt} \right\rangle = Q \langle v_x B \rangle \hat{z} = \frac{Q}{c} \langle v_x E \rangle \hat{z}
\]

where \( P \) is the charge’s momentum.

- Meanwhile, the rate at which kinetic energy \( K \) is imparted by the wave to the charge is

\[
\frac{dK}{dt} = v \cdot F = Qv \cdot E + Qv \cdot (v \times B) = Qv \cdot E
\]

\[
= Qv_x E \quad , \quad \text{or}
\]

\[
\left\langle \frac{dK}{dt} \right\rangle = Q \langle v_x E \rangle
\]
Momentum (continued)

Thus

\[ \langle \frac{dP}{dt} \rangle = \frac{1}{c} \langle \frac{dK}{dt} \rangle \hat{z} \]

That is, if motion of charges removes energy from the wave at the rate \( \langle dK/dt \rangle \), it removes momentum at the same rate.

And so if the energy per unit time and area carried by the wave is

\[ S = \frac{E \times B}{\mu_0} , \]

the momentum per unit time and area carried by the wave is \( 1/c \) of that, or

\[ \mathcal{P} = \frac{S}{c} = \sqrt{\frac{\varepsilon_0}{\mu_0}} E \times B , \]

same as the quantum result.
Radiation pressure

And so electromagnetic waves can exert a force or pressure in the direction they travel.

- Consider, for example sunlight a distance $r$ away from the Sun. Sunlight itself, emitted uniformly in all directions:
  
  Total power output from the Sun: $L = L_\odot = 3.85 \times 10^{26} \text{ W}$
  
  Energy per unit time and area at $r$: $S = L/4\pi r^2$
  
  Momentum per unit time and area: $\mathcal{P} = \hat{r} L/4\pi cr^2$

- Suppose the sunlight is incident normally upon a body with surface area $\sigma$, which reflects a fraction $A$ of the incident light and absorbs the rest.
  
  - $A$ is called the **albedo** of the body.
Radiation pressure (continued)

Then, since momentum is conserved, the force on the body is

\[ F = \frac{dP}{dt} = (1 - A) \sigma \mathcal{P} + 2A \sigma \mathcal{P} \]

\[ = (1 + A) \frac{L \sigma}{4\pi cr^2} \hat{r} \]

The radial force per unit area, \( F/\sigma \), is called the radiation pressure. Like any other pressure, its SI units are newtons per square meter.

Note that the force on a perfectly reflecting body \( (A = 1) \) is twice as large as that on a perfectly absorbing body \( (A = 0) \).
Radiation pressure (continued)

Example. The International Space Station has a mass of $m_{\text{ISS}} = 2.27 \times 10^5$ kg, and is in orbit a distance $a = 6738$ km from the Earth’s center (altitude 360 km). Design a “solar sail” which could carry the ISS out of the Solar system.

The maximum gravitational force on the ISS is that exerted on it when it’s on the opposite side of the Earth from the Sun:

$$F_{\text{max}} = - \frac{GM_E m_{\text{ISS}}}{a^2} - \frac{GM_{\odot} m_{\text{ISS}}}{(r + a)^2}$$

where $r = 1$ AU $= 1.5 \times 10^{11}$ m and $M_{\odot} = 2 \times 10^{30}$ kg.
Radiation pressure (continued)

☐ The radiation force must offset this. We need to find the area $\sigma$ which makes this work:

$$ F = (1 + A) \frac{L\sigma}{4\pi cr^2} \hat{r} > -F_{\text{max}} $$

$$ (1 + A) \frac{L\sigma}{4\pi cr^2} > \frac{GM_E m_{\text{ISS}}}{a^2} + \frac{GM_{\odot} m_{\text{ISS}}}{(r + a)^2} $$

$$ \sigma > \frac{4\pi cr^2}{L(1 + A)} \left( \frac{GM_E m_{\text{ISS}}}{a^2} + \frac{GM_{\odot} m_{\text{ISS}}}{(r + a)^2} \right) $$

$$ > 2.2 \times 10^{11} \text{ m}^2 \text{ for } A = 1 $$

☐ If square, it’s 469 km on a side.
Modulated waves

The plane waves we have considered so far actually do not transmit anything.

- We have assumed that they are infinite in extent – how could they?

So we have implicitly had in mind turning the waves on and off, or equivalently getting in a wave’s way all of a sudden (e.g. by deploying a solar sail at some instant in time).

- Turning a wave on and off is an example of wave modulation (in this case amplitude modulation [AM]).

- It is by modulation that electromagnetic waves can be used to send signals, and to transfer energy and/or momentum through vacuum.
Modulated waves (continued)

Modulation is produced by having the generator of the wave combine two waves with different frequencies.

- Usually the carrier frequency is lots larger than the modulation frequency.
- Simply adding the fields algebraically produces amplitude modulation.
Modulated waves (continued)

- For plane waves propagating in vacuum, it can be shown (fairly easily) that the modulations travel at the same speed as the peaks and troughs of the waves that make them up.
  - The speed of modulation propagation is called the **group velocity**, and that of the component peaks and troughs the **phase velocity**.
- Waves propagating in matter needn’t obey this relation.
  - Phase velocity can under some conditions exceed the speed of light in vacuum.
  - Group velocity cannot, though. The ultimate speed at which a **signal** can travel is the speed of light in vacuum.