Today in Physics 122: Examples in review

By class vote:

- Problem 22-40: off-center charged cylinders
- Problem 28-39: $B$ along axis of spinning, charged disk
- Problem 30-74: self-inductance of a toroid.

Double rainbow over the VLA. If you can explain everything in this picture, you will have understood all of electricity and magnetism.
Problem 22-40

A very long nonconducting cylinder of radius $R_1$ is uniformly charged with charge density $\rho_E$. It is surrounded by a cylindrical metal (conducting) tube of inner radius $R_2$ and outer radius $R_3$, which has no net charge. If the axes of the two cylinders are parallel, but displaced from each other by a distance $d$, determine the resulting electric field in the region $R > R_3$, where the radial distance $r$ is measured from the metal cylinder’s axis. Assume $d < (R_2 - R_1)$. 
Problem 22-40 (continued)

The key features of this problem are that

(a) it involves a superposition of two infinite cylindrical charge distributions, so the electric field has no component in the direction of the axes, suggesting the use of Gauss’s Law; and

(b) \( E = 0 \) inside the conducting cylindrical shell, also suggesting the use of Gauss’s Law.
Problem 22-40 (continued)

- The charge per unit length of the nonconducting cylinder is
  \[ \lambda_1 = \rho_E \pi R_1^2. \]
  Call that on the other surfaces \( \lambda_2 \) and \( \lambda_3 \).

- Draw a Gaussian cylinder, radius \( r \) and length \( h \), coaxial with the conducting shell and lying within it. \( E = 0 \) everywhere on its surface, except the circular ends, which are \( \perp E \) (i.e. \( E \cdot dA = 0 \)).
Problem 22-40 (continued)

- So Gauss’s Law for this surface is

\[ \oint E \cdot dA = 0 = \frac{1}{\varepsilon_0} Q_{encl} = \frac{(\lambda_1 + \lambda_2) h}{\varepsilon_0} \]

\[ \Rightarrow \lambda_2 = -\lambda_1 = -\pi \rho_E R_1^2 \]

- The charge/length \( \lambda_2 \) is distributed nonuniformly on the inner surface of the shell, in such a way that its \( E \) cancels that of the charged rod, within the bounds of the shell.

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Gaussian cylinders of length \( h \), perpendicular to the page.
Problem 22-40 (continued)

- The shell has no net charge, so
  \[ \lambda_3 = -\lambda_2 = \pi \rho_E R_1^2. \]

- This charge must be distributed uniformly on the outer surface.
  - Otherwise its \( E \) would not be zero inside the shell, as it must be since \( E \) from the other two charge distributions add up to zero within the shell.
Thus the calculation for the field outside the shell is familiar:

\[ \oint E \cdot dA = \frac{Q_{\text{encl}}}{\varepsilon_0} \]

\[ E2\pi Rh = \lambda_3 h/\varepsilon_0 = \pi \rho_E R_1^2 h/\varepsilon_0 \]

\[ E = \frac{\pi \rho_E R_1^2}{2\pi \varepsilon_0} \hat{r} \quad . \]

Note that the answer does not depend on where the charge rod is, as long as it’s inside the shell.
Problem 28-39

A nonconducting circular disk, of radius $R$, carries a uniformly-distributed electric charge $Q$. The plate is set spinning with angular velocity $\omega$ about an axis perpendicular to the plate through its center. Determine

(a) its magnetic dipole moment and

(b) the magnetic field on its axis a distance $z$ away from the center.

(c) Is $B \approx \frac{\mu_0}{2\pi} \frac{\mu}{z^3}$ for $z \gg R$?

I’ll do (b) first.
Problem 28-39 (continued)

(b) We first break the problem down into parts we have seen before.

- Consider an infinitesimal annulus, \( dr \) wide at radius \( r \).
- This annulus carries a charge

\[
\,dQ = \sigma dA = \sigma 2\pi rdr,
\]

and the disk is spinning, so the annulus is like a circular loop of current:

\[
dI = \frac{dQ}{\tau} = \frac{\omega}{2\pi} dQ = \omega \sigma rdr
\]

Charge \( Q \), radius \( R \)

Charge density \( \sigma = Q / \pi R^2 \)
That is a problem we’ve seen before: in class on 30 October 2012 we showed that the magnetic field on the axis of a circular loop of radius \( R \) that carries a current \( I \) is

\[
B = \frac{\mu_0}{2} \frac{R^2 I}{\left( R^2 + z^2 \right)^{3/2}} \hat{z}
\]

So switch \( I \rightarrow dI, R \rightarrow r : \)

\[
dB = \frac{\mu_0}{2} \frac{r^2 dI}{\left( r^2 + z^2 \right)^{3/2}} \hat{z} = \frac{\mu_0 \omega \sigma}{2} \frac{r^3 dr}{\left( r^2 + z^2 \right)^{3/2}} \hat{z}
\]
Problem 28-39 (continued)

- And so we add the contributions of all the annuli into which the disk can be decomposed:

\[
B = \hat{z} \frac{\mu_0 \omega \sigma}{2} \int_0^R \frac{r^3 dr}{(r^2 + z^2)^{3/2}}
\]

- Substitute:

\[
u = r^2 + z^2 \quad du = 2rdr
\]

As \( r = 0 \to R, u = z^2 \to R^2 + z^2 \), so...
Problem 28-39 (continued)

\[
B = \hat{z} \frac{\mu_0 \omega \sigma}{4} \int_0^R \frac{r^2 2rdr}{\left(r^2 + z^2\right)^{3/2}} = \hat{z} \frac{\mu_0 \omega \sigma}{4} \int_{z^2}^{R^2 + z^2} \frac{(u - z^2)du}{u^{3/2}}
\]

\[
= \hat{z} \frac{\mu_0 \omega \sigma}{4} \left[ \frac{u^{1/2}}{1/2} - z^2 \frac{u^{-1/2}}{-1/2} \right]_{z^2}^{R^2 + z^2}
\]

\[
= \hat{z} \frac{\mu_0 \omega \sigma}{2} \left[ \sqrt{R^2 + z^2} - z + \frac{z^2}{\sqrt{R^2 + z^2}} - z \right]
\]

\[
= \hat{z} \frac{\mu_0 \omega \sigma z}{2} \left[ \sqrt{1 + \left(\frac{R}{z}\right)^2} - 2 + \frac{1}{\sqrt{1 + \left(\frac{R}{z}\right)^2}} \right]
\]
(a) The magnetic dipole moment of the infinitesimal annulus at radius $r$ is its current times its enclosed area:

$$d\mu = \hat{z}Adl = \hat{z}\pi\omega\sigma r^3 dr$$

so the sum of the moments of all the annuli is just the integral of this:

$$\mu = \hat{z}\pi\omega\sigma \int_0^R r^3 dr = \hat{z}\pi\omega\sigma \frac{R^4}{4}$$
Problem 28-39 (continued)

(c) For this we need a power-series expansion that moves the problem past the scope of PHY 122:

\[ (1 + x)^n = \sum_{m=0}^{\infty} \frac{n!}{m!(n-m)!} x^m \]

\[ \approx 1 + nx + \frac{n(n-1)}{2} x^2 \quad \text{if } |x| \ll 1. \]

Here we have in mind \( x = (R/z)^2 \ll 1 \) and \( n = \pm 1/2 \):

\[ B = \frac{\mu_0 \omega \sigma z}{2} \left[ \left( 1 + (R/z)^2 \right)^{1/2} - 2 + \left( 1 + (R/z)^2 \right)^{-1/2} \right] \]
Problem 28-39 (continued)

\[ B = \hat{z} \frac{\mu_0 \omega \sigma z}{2} \left[ \left( 1 + \left( \frac{R}{z} \right)^2 \right)^{1/2} - 2 + \left( 1 + \left( \frac{R}{z} \right)^2 \right)^{-1/2} \right] \]

\[ = \hat{z} \frac{\mu_0 z}{2} \left( \frac{4 \mu}{\pi R^4} \right) \left[ 1 + \frac{1}{2} \left( \frac{R}{z} \right)^2 - \frac{1}{8} \left( \frac{R}{z} \right)^4 - 2 + 1 - \frac{1}{2} \left( \frac{R}{z} \right)^2 + \frac{3}{8} \left( \frac{R}{z} \right)^4 \right] \]

\[ = \frac{\mu_0 \mu}{2 \pi z^3} \hat{z} \]

So, \text{yeah,} it does what is suggested in the book.
(a) Show that the self-inductance \( L \) of a toroid of radius \( r_0 \) containing \( N \) loops each of diameter \( d \) is

\[
L \approx \frac{\mu_0 N^2 d^2}{8r_0} \quad \text{if } r_0 \gg d.
\]

(b) Calculate \( L \) for \( d = 2.0 \text{ cm} \) and \( r_0 = 66 \text{ cm} \). Assume that the field inside the toroid is uniform, and that there are 550 loops in it.
Let’s calculate the field inside the toroid first. The circular geometry will force $B$ to be constant in magnitude on circles and point clockwise, so we use Ampère’s Law and a circular loop:

\[ \oint B \cdot d\ell = \mu_0 I_{\text{encl}} \]

\[ B2\pi r = \mu_0 NI \]

\[ B = \frac{\mu_0 NI}{2\pi r} \left( -\hat{\phi} \right) \]
Problem 30-74 (continued)

- If \( d \ll r_0 \) then all points within the toroid lie at approximately the same radius: the field is almost uniform.

- Thus the self-inductance becomes

\[
L = \frac{N \Phi_B}{I} \simeq \frac{NBA}{I} = \frac{\mu_0 N^2 (d/2)^2}{2\pi r_0} \pi \left( \frac{d}{2} \right)^2 = \frac{\mu_0 N^2 d^2}{8r_0}
\]

as advertised.
Problem 30-74 (continued)

- The arithmetic:

\[ L = 2.9 \times 10^{-5} \text{ H.} \]