Summary of Research and Plans for the Future

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A substantial part of my research is aimed at a better understanding of the dynamics of Yang–Mills theories. The work spans many years and is branching out into various related areas of mathematical physics.

1 *Why study Yang–Mills Theory?*

Quantum Field Theories have proven to be the fundamental paradigm of high energy physics and much of many body theory as well [1]. The early spectacular successes of Quantum Electrodynamics focused attention on the importance of gauge invariance. With the twin revolutions of the seventies in our understanding of high energy physics- the electro-weak theory and quantum chromo-dynamics- it is unquestioned that the most fundamental explanation of the material world to date is in terms of non-abelian Yang-Mills theories. It remains important to understand these theories at a deeper level even as we search for more unified theories.

In the electro-weak sector, interactions are amenable to perturbation theory at energies below a few TeV. Hence there is a satisfactory method for computation of all interesting physical quantities. (If the theory remains unchanged at high energies, without super-symmetry or grand unification, perturbation theory could fail at high energies.)

However, in the arena of strong interactions, the successes were not as complete: it is at low energies that perturbation theory breaks down. Thus even though we know the fundamental lagrangian, we have to struggle to derive properties of observable particles from it. This battle continues and is likely to go on for quite a while. A historical analogue would be celestial mechanics, where the fundamental equations of motion was known but classical perturbation theory failed to provide a complete description of long time behavior. Indeed almost all of the mathematical techniques of modern physics-and much of mathematics- were developed originally to solve problems in celestial mechanics.

In much the same way a deeper understanding of the Yang-Mills theory remains the fundamental theoretical challenge of modern particle physics. An alternative to the perturbative expansion was suggested by 't Hooft[2] (the $\frac{1}{N}$ expansion) but even the leading order of this has proved to be intractable so far. It seems necessary to work these ideas out in simpler contexts ('matrix models') and to build bridges with ideas in mathematics ('non-commutative probability theory') before we can work out the realistic cases. This has been the main thrust of my research in the last few years.

It is unlikely that Quantum Chromodynamics is an exactly solvable theory: very few interesting physical systems are solvable in any case. Thus it is important to have approximation methods that will capture the essence of the dynamics and which allow for a systematic improvement. The most promising such idea for QCD is that of the $\frac{1}{N}$ expansion proposed by 't Hooft and studied by many other authors since then.

In the realistic QCD, the basic invariance group is $SU(3)$. If we replace this by a theory invariant under $SU(N)$, something remarkable happens: the quantum fluctuations in gauge invariant observables is of order $\frac{\hbar}{N}$. Thus we get a classical theory not in the limit $\hbar \to 0$ keeping N fixed (at three for example) but also in the new limit $N \to \infty$ keeping $\hbar$ fixed. This new classical limit is however looks entirely different from Yang-Mills theory. It is not a local quantum field theory for example. The 'quantization' of this new classical theory will restore all corrections of order $\frac{1}{N}$. All of this has been demonstrated in two space-time dimensions.

2 *Connections to String Theory*

There is a wealth of old experimental data on strong interactions and there is promise of new data from the Jefferson Lab for example. The new ideas on string theory could also shed light on QCD: there is some reason to believe that string theory could provide a theory of strong interactions (independent of whether it is a theory of quantum gravity). The existence of a large (surely infinite) number of excited states, the Regge behavior, the mysterious phenomenon of quark-hadron duality all pint to this. Also, there are deep theoretical ideas of Maldacena[3] which point to an equivalence of a string theory with a field theory at least in the limit of exact conformal invariance. We could learn much about such equivalences by studying toy models, for example in lower space-time dimensions.

With Abhishek Agarwal (graduate student) I have recently looked [4] at the dilatation operator of N=4 Super Yang–Mills theory. This turns to be the hamiltonian of a matrix model of the type I had studied some time ago. Using an equivalence of the large N limit of matrix models with quantum spin chains I had established some years ago [5] we were able to clarify the integrability of this limit discovered by Minahan, Zarembo and others. We found that the Poisson algebra of the large N limit gives a good framework in which to understand the conserved charges of this system. This is the approach we will be concentrating on for the immediate future.

3 *Hadrons in Two Space-Time Dimensions and Beyond*

Some years ago, I worked out the complete theory of hadrons in the special case of two dimensional space-time [6]. This was an essential step towards constructing a theory of hadrons in the real world. 't Hooft had only understood the meson spectrum. I constructed a theory of hadrons including baryons as well a solitons, connecting the ideas of 't Hooft and Witten with earlier ideas of Skyrme. Also, the resulting hadronic theory turns out to be an open string field theory. The essential simplifying feature of two dimensions is that the gluons can be eliminated as degrees of freedom, resulting in a theory of quarks interacting through a potential. I have used this limiting case of QCD as a approximation to the Deep Inelastic limit of QCD and understood some features of the quark structure functions of the nucleons. Much work still needs to be done in this direction.

The hadronic theory we seek in higher dimensions is a kind of field theory whose basic dynamical variable is a function on the space of closed curves in space-time: loop space. (I have in the past studied the basic geometry of this space in a different, string theoretic context.) New mathematical ideas are needed to formulate such a theory (a kind of string field theory). It turns out that many such tools were developed by K. T. Chen[9] , in his work on the homotopy theory of loop spaces. There are also close connections with modern ideas on the braid group that we intend to follow up.

4 *Yang–Mills on 2+1 Dimensions*

There are now results on three dimensional Yang–Mills theory due to Karabali, Kim and Nair [7] which provide a new impetus to this program. They have essentially proved confinement in 2+1 dimensional Yang-Mills theory and even computed the string tension exactly. Moreover, they have shown that certain correlation functions decay exponentially. Their work does not shed much light of the spectrum of hadrons in 2+1 dimensional QCD.

We have computed [8] the mass gap (the difference between the two lowest eigenvalues of the hamiltonian) of 2+1 non-abelian Yang-Mills theory exactly. If we split the hamiltonian of $2+1$ QCD into $H = H_0 + H_1$ where $H_0 = \frac{1}{2}\alpha \int$ tr $E^2 d^2x$ and $H_1 = \frac{1}{2\alpha} \int$ tr $B^2 d^2x$ and ignore H_1, we get a reduced Yang-Mills theory which is gauge invariant and has an additional conformal invariance. We find a version of the lattice regularization (using simplicial sets) that preserves a discrete version of this conformal invariance and solve the reduced theory exactly. The continuum limit can be taken by taking advantage of this discrete conformal invariance. The spectrum consists of 'glue-rings' whose energy is proportional to their length. The magnetic energy H_1 can then be added as a perturbation; to second order it describes non-relativistic motion of the glue-rings. The mass gap of 2+1 Yang-Mills theory is not corrected to any order in this perturbation theory. 2+1 dimensional QCD is the limiting case of QCD at high temperatures: hence it could be relevant to understanding the quark gluon plasma observed at the RHIC accelarator.

5 *Matrix Models: Noncommutative Geometry and Probability Theory*

We can currently realize these ideas completely on theories with N degrees of freedom, such as two dimensional QCD (because the gluons can be eliminated). When the degrees of freedom are matrices (i.e., N^2 per spacetime point) as with gluons, we need new methods. A necessary step is to understand quantum system described by a finite number of $N \times N$ matrices. The case of one such matrix degree of freedom is well-known. Much work on multimatrix models exist in the context of string theory, but certain crucial ingredients like variational principles are missing.

A substantial part of our work in the last two years has been to work out the large N limit of matrix models. In the large N limit, these theories do reduce to 'classical limits' whose 'equations of motion' are an infinite system of algebraic equations (often called the Migdal-Makeenko or 'factorized Schwinger-Dyson' equations.) These equations cannot usually be solved exactly. It would not be sufficient to solve them in the perturbative expansion, except as a check that we have the right large N limit. A new approximaton method is needed.

We found that there is a variational principle from which these equations could be derived. Such a variational principle should exist if the basic idea that the large N limit is a classical theory is correct. Nevertheless its constuction involves a subtelty: when we reformulate the theory in terms of gauge invariant variables, an entropy needed to be added. (We related these physical ideas to the work of D. V. Voiculescu on entropy of states in operator algebras [10].)

Moreover this entropy cannot be expressed as an algebaric expression in terms of the gauge invariant variables: it is a kind of anomaly. Nevertheless we found expressions for this quantity. It allows us to make variational approximations to the ground state of many hitherto unsolvable matrix models. These solutions are in agreement with exact solutions in the few cases where they are available.

Moreover, we have performed Monte-Carlo simulations of other matrix models which also are also in agreement with our results. Thus we believe we have found a new class of techniques to attack the large N limit of gauge theories. The main obstacle remaining is to carry out our analysis in the presence of an infinite number of matrix degrees of freedom. We have made some progress in this direction.

6 *A Theory of Errors in Quantum Measurement*

As another spinoff of our matrix model work, we obtained[11] a theory of errors in quantum measurement. The main idea is that in measuring an observable (represented by a hermitean matrix A) there is inevitably some random error B added to it. The probability distribution of the sum $A + B$ can be computed using techniques familiar to us from our work on random matrix theory. The answer should be verifiable experimentally.

7 *Schrödinger operators with point interactions in the plane*

It has been known for a long time that attractions of zero range (point-like) of non-relativistic particles provide a simple model of asymptotic freedom. We develop [12] a non-perturbative renormalization method in this system which predicts the bound states as well as the scattering amplitude of the system in terms of one fundamental energy scale. In order to establish that asymptotically free renormalizable theories such as this one have a perfectly finite formulation, we establish a theorem (using techniques from functional analysis) that the hamiltonian is self-adjoint and bounded below after renormalization. These results hold not only to all orders in perturbation theory, but even include effects that perturbation theory cannot describe at any order (such as the formation of bound states.)

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