Chapter 3

Determinants

3.1 We will now see how to extend the notion of a gaussian integral to the case of an infinite number of variables. Clearly this is going to involve the notion of a determinant of an infinite dimensional matrix. One way to interpret such a matrix is as an operator in a Hilbert space.

3.2 The *norm* of an operator A is

$$|A| = \sup u \frac{|Au|}{|u|}.$$

Operators for which it exists are said to be *bounded*; the set of bounded operators $\mathcal{B}$ is an algebra.

3.3 The rank of an operator $A : \mathcal{H} \to \mathcal{H}$ is the dimension of its image. Of couse finite rank operators are bounded; the subset $\mathcal{I}^0$ of finite rank operator is a two-sided ideal in $\mathcal{B}$. The completion of $\mathcal{I}^0$ is however not $\mathcal{B}$; it is the space of *compact* operators $\mathcal{I}$. It is a proper two-sided ideal of $\mathcal{B}$.

3.4 Any finite rank operator can be expanded in terms of its *characteristic values* a_i and its left and right *eigenvectors* psi_i and chi_i respectively: $A = \sum_i a_i \psi_i \otimes \chi_i^\dagger$. The number of non-vanishing terms in this series is the rank.

3.5 There are several more fine norms on the space of finite rank operators,

$$|A|_p = \left[\sum_i |a_i|^p\right]^{\frac{1}{p}}, \quad \text{for } p \geq 1.$$

We can regard the previous norm as the limiting case $p \to \infty$. The completion of the space of finite rank operators by this norm gives a two sided ideal of $\mathcal{I}^p = \mathcal{B}$. These are non-commutative analogues of the Banach spaces L^p of a measure. For example,

$$\mathcal{I}^p \mathcal{I}^q \subset \mathcal{I}^r, \text{ for } \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \quad \mathcal{I}^p \subset \mathcal{I}^q, \text{for } p < q.$$

The special cases $p = 1$ and $p = 2$ are of interest; they are respectively the space of *trace-class* operators and *Hilbert-Schmidt* operators.

3.6 Suppose that A is trace-class; i.e., $\sum_i |a_i|$ is convergent. Then the determinant can be defined to be the product:

$$\det K = \prod_i [1 + a_i].$$

Recall (eg., Whittacker and Watson) that an infinite product $\prod_i (1 + a_i)$ is absolutely convergent iff the sum $\sum_i |a_i|$ is convergent, and if only a finite number of the factors is zero.

3.7 This determinant has all the familiar properties of finite dimensional determinants:

$$\det K_1 K_2 = \det K_1 \det K_2, \quad \text{for} K_1 - 1, K_2 - 1 \in \mathcal{I}^1$$

$$\det S K S^{-1} = \det K, \text{ for } K - 1 \in \mathcal{I}^1, S \in \mathcal{B}$$

Also the eigenvalues of K are the zeros of the entire function

$$f(z) = \det [K - z] .$$

This determinant is an entire function of the matrix elements of the operator K .

3.8 What if A is not trace-class? Suppose $\mathrm{tr} A^\dagger A < \infty$; i.e., A is *Hilbert-Schmidt*. Then $e^{-A}(1 + A) - 1$ is trace-class and we can define a modified determinant

$$\det_1 (1 + A) = \det \left[e^{-A}(1 + A) \right] .$$

Of course, if A is trace-class,

$$\det_1(1+A) = e^{-\operatorname{tr} A} \det[1+A]$$

Thus it can be thought of as a *renormalization* of the usual determinant. It does not have the multiplicative property:

$$\det_1 K_1 K_2 \neq \det_1 K_1 \det_1 K_2.$$

3.9 More generally, for $A \in \mathcal{I}^p$ with integer p, we put

$$\det_p[1+A] = \det\left[e^{\sum_{k=1}^{p}(-1)^k \frac{A^k}{k}}(1+A)\right]$$