# Chapter 1

# Moments and Effective Actions

**1.1** We will be studying probability distributions of the form $\frac{1}{Z_0}e^{-S(\phi)}d^M\phi$ on $R^M$ . The function $S : R^M \to R$ (called the *action*) is assumed to be bounded below; also it goes to infinity as $\phi \to \infty$ fast enough to make the integral converge. In most cases of interest $S$ is a polynomial. The constant $Z_0$ is chosen so that $\frac{1}{Z_0}\int e^{-S(\phi)}d^M\phi = 1$ .

**1.2** The *moments* of a probability distribution are the expectation values

$$G^{i_1\cdots i_n} =< \phi_1^i \cdots \phi^{i_n} >:= \frac{1}{Z_0}\int \phi^{i_1}\cdots\phi^{i_n}e^{-S(\phi)}d^M\phi.$$

They are also called *n-point functions*. The *characteristic function* of a probability distribution is its Fourier transform:

$$Z(J) =< e^{iJ^T\phi} >$$

The characteristic function is the generating function for the moments:

$$G^{i_1\cdots i_r} = \frac{1}{i^n}\left[\frac{\partial^n Z(J)}{\partial J_{i_1}\cdots\partial J_{i_n}}\right]_{J=0}$$

**1.3** The negative logarithm $W(J)$ of the characteristic function is called the *generating function of connected moments*; the *connected moments* are therefore defined by

$$e^{-W(J)} =< e^{iJ^T\phi} >, \quad G_c^{i_1\cdots i_r} = -\frac{1}{i^n}\left[\frac{\partial^n W(J)}{\partial J_{i_1}\cdots\partial J_{i_n}}\right]_{J=0}.$$

**1.4** We can derive a formula for the moments in terms of the connected moments, from the relation between their generating functions. This will also justify the name 'connected moments':

$$G^{i_1\cdots i_n} = \sum_{r=1}^{n} \sum_{\{S_1, S_2, \cdots S_r\} \in \mathcal{P}_r(\{1,\cdots n\})} G_c^{i_{S_1}} \cdots G_c^{i_{S_r}}.$$

Here, $\mathcal{P}_r(S)$ denotes the set of partitions of $S = \{1, \cdots, n\}$ into $r$ subsets; i.e., an element of $\mathcal{P}_r(S)$ is a set of nonempty subsets $S_1 \cdots S_r$ of $S$ such that $S_a \cap S_b = \phi$ for $a \neq b$ and $\bigcup_{a=1}^{r} S_a = S$ . Also, $G_c^{i_{S_a}}$ denotes the connected moment with indices labeled by the numbers in the subset $S_a$ ; since the connected moments are symmetric, only the subset matters, not the order in which the indices appear.

**1.5** To prove this, note that

$$W(J) \sim -\sum_{n=0} G_c^{i_1\cdots i_n} \frac{iJ^{i_1} \cdots iJ^{i_n}}{n!}, \quad e^{-W(J)} = Z(J) \sim \sum_{n=0} G^{i_1\cdots i_n} \frac{iJ^{i_1} \cdots iJ^{i_n}}{n!}.$$

The tilde means that we are to think of the r.h.s. as formal power series expansions, not as convergent series. By substituting the first series into the second we can prove the relation between connected moments and moments.

**1.6** The *effective action* $\Gamma(\Phi)$ is defined as the Legendre transform of $W$ :

$$\Gamma[\Phi] = \min_J \left[ W(J) + iJ^T\Phi \right].$$

For this to make sense, we assume that $W(J)$ is a *convex function*; i.e., that the second derivative of $W(J)$ is a positive matrix for all $J$ . It follows that $W(J)$ is the inverse Legendre transform of $\Gamma(\Phi)$ :

$$W[J] = \min_\Phi \left[ \Gamma(\Phi) - iJ^T\Phi \right].$$

**1.7** The effective action is the generating function of the *effective vertices*:

$$\Gamma_{i_i\cdots i_n} = \left[ \frac{\partial^n \Gamma(\Phi)}{\partial\Phi^{i_1} \cdots \partial\Phi^{i_n}} \right]_{\Phi=\Phi_0}$$

where $\Phi_0$ is the minimum of $\Gamma$. Since we are assuming that $W(J)$ is convex, so is $\Gamma(\Phi)$ and hence the minimum is unique.

**1.8** The effective action determines $W(J)$ and hence $Z(J)$. So it should not be surprising that the effective vertices determine the moments. First the connected moments are given in terms of the effective vertices as follows:

$$G_c^i = \Phi_0^i, \qquad \Gamma_{ij}G_c^{jk} = \delta_i^k \quad \text{and } G_c^{i_1\cdots i_n} = \Gamma_{j_1\cdots j_n}G_c^{i_1j_1}\cdots G_c^{i_nj_n}, \text{ for } n \geq 3.$$

The moments are then given by the above formula in terms of the connected moments.

**1.9** We can also verify the following expression:

$$e^{-\Gamma(\Phi)} = \frac{1}{Z_0}\int e^{-\left[S(\Phi+\phi)-\phi^i\frac{\partial\Gamma}{\partial\phi^i}(\Phi)\right]}d^M\phi.$$

This is the starting point of some approximation schemes to evaluate the effective action.

**1.10** *Example*: If $S(\phi) = \frac{1}{2}\phi^T A\phi$ is a positive quadratic form, we get a *Gaussian* distribution. The normalization constant can be determined by diagonalizing $A$ so that the probability measure is $\frac{\sqrt{\det A}}{(2\pi)^{\frac{n}{2}}}e^{-\frac{1}{2}\phi^T A\phi}d^n\phi$. Also, $A^{-1}$ is called the *covariance matrix*.

**1.11** For the Gaussian, $Z(J) = e^{-\frac{1}{2}J^TA^{-1}J}$. Thus $W(J) = \frac{1}{2}J^TA^{-1}J$, $\Gamma(\Phi) = S(\Phi)$.

**1.12** Thus, the only connected moment that is non-zero is the second one:

$$< \phi^{i_1}\phi^{i_2} >_c = < \phi^{i_1}\phi^{i_2} > = [A^{-1}]^{i_1i_2}.$$

The moments of odd order are zero. The even moments are given by *Wick's theorem*, a special case of the earlier formula in terms of the connected moments:

$$< \phi^{i_1}\cdots\phi^{i_{2r}} > = \sum_{\text{pairings}}\prod_{\text{pairs } ab}[A^{-1}]^{i_ai_b}$$

**1.13** We can also state the above answer in graphical terms: put down a set of points labelled $i_1, i_2, \cdots i_{2r}$. Connect them pairwise by (unoriented) lines; to each line connecting $i_a$ and $i_b$ we have a factor $[A^{-1}]^{i_ai_b}$. We then sum over all such diagrams.