Lecture on Branched Polymers and Dimensional Reduction

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Abstract

This is a pedagogical account of the some of the recent results of Brydges and Imbrie, described from the point of view of Grassmann integration. Some simple extensions are pointed out.

Historical Introduction

The subject of dimensional reduction, in the context considered in this lecture, has a long and rather peculiar history. The strangest aspect of all is that the simplest formulation of the results has been found only very recently.

That one physical theory in $d$ space dimensions should be related to another in $D = d - 2$ dimensions was first suggested in the context of the field-theoretic formulation of the critical behaviour of the Ising model in a quenched random magnetic field [1,2]. It was noticed that the most singular Feynman diagrams all had the form of trees before averaging over the quenched randomness. When this was done, it turned out that the diagrams were the same as those for the critical Ising model without any random field, in two fewer dimensions.

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Thus, no further work was needed – one should be able to take, for example, the known results for the ordinary Ising model in $D$ dimensions, and find the critical behaviour of the random field Ising model in $d = D + 2$ dimensions. In particular, since the lower critical dimension (at and below which there is no phase transition) of the ordinary Ising model is $D = 1$, this should imply that the 3-dimensional random field Ising model has no phase transition.

Unfortunately this beautiful idea is wrong. In particular it contradicts a very simple argument of Imry and Ma [1] that the lower critical dimension should be two. However, it took several years of theoretical and experimental confusion and contradictory papers before Imbrie [3] proved that Imry and Ma were right.

During this time Parisi and Sourlas [4] came up with a startling explanation of how and why dimensional reduction works: supersymmetry. At that time supersymmetry was thought to be the preserve of particle physics and string theory, so their argument was not widely understood. In their paper was also the germ of the reason why simple dimensional reduction might not work for the random field Ising model: the tree diagrams were seen to be the perturbative solution to the classical field equations in the presence of the random external field. If the solution of these equations was unique and given by the sum of the diagrams, then all should have been well. But it was easy to see that, at low enough temperatures, the solution was not unique, and that the one with the lowest energy was not the perturbative one. Work is still continuing on how, possibly, to rectify this situation, but this seems a very difficult problem.

Meanwhile, a few years later, Parisi and Sourlas [5] came up with another ingenious application of their ideas. The problem known under various versions as branched polymers, lattice trees, or lattice animals (to be described later) had been attracting some attention. Lubensky and Isaacson [6] had formulated a rather complicated field theory for this problem, had come to the conclusion that the upper critical dimension (above which mean field theory is valid) is $d = 8$, and had noticed that the first terms in the $\epsilon$-expansion of the critical exponents below this were the same as those of the so-called Yang-Lee edge singularity
in $D = d - 2$ dimensions. This is the problem of an Ising model in a purely imaginary magnetic field, and is described by a simple scalar field theory with a cubic interaction and a purely imaginary coupling. Parisi and Sourlas reformulated the model of Lubensky and Isaacson in a simpler way, and then used their supersymmetry to explain why the dimensional reduction happened. This time, everything was all right. The numerically measured exponents of lattice animals and lattice trees in $d = 3$ dimensions are given by the exactly known Yang-Lee exponents in $D = 1$. A non-perturbative proof of dimensional reduction in a supersymmetric field theory was given [7]. It was understood that the main reason for the failure of dimensional reduction in the random field Ising model does not apply here, essentially because the tree diagrams are the branched polymer configurations [8]. But none of these arguments were completely watertight, largely because it seemed necessary to ignore so-called irrelevant terms which spoiled the supersymmetry, but were believed not to affect the critical behaviour.

In fact, the Yang-Lee critical theory is nowadays strongly believed [9] to be in the same universality class as an even simpler problem: a classical gas with short-range repulsive interactions in the grand canonical ensemble. As a function of the fugacity $z > 0$, the grand partition function is sum of positive terms and has no singularity nor any zeroes. But in the complex $z$ plane the closest singularity to the origin lies on the negative axis at $z = -z_c$, and the critical behaviour near this is believed to be independent of the particular details of the interaction (so it is known as the ‘universal repulsive gas singularity’), and it is also believed to be the same as that of the Yang-Lee problem at the critical imaginary magnetic field. Thus the branched polymer problem in $d$ dimensions was believed to be related to the universal repulsive gas singularity in $D = d - 2$ dimensions. But once again these arguments relied on the neglect of irrelevant terms in the appropriate field theories.

It took Brydges and Imbrie [10] to realise that most of this theoretical undergrowth could be cut away - that field theory is not needed at all! There is a simple and physical model for branched polymers in $d$ dimensions for which dimensional reduction to a repulsive gas is rigorous and exact, not just in the critical region (which in this context means very large
polymers) but for all values of the fugacity. The arguments do use supersymmetry, in a very beautiful way, and give compelling evidence of how important it is in theoretical physics to learn how to walk (do problems with finitely many particles) before one tries to run (field theory and beyond.)

The aim of these lectures is to present these ideas in as simple a manner as possible, using the ideas of Grassmann integration nowadays familiar to many theoretical physicists. I shall, however, quickly recall all the ingredients necessary for the argument.

**Classical gas and cluster expansion.**

Consider a classical gas of point particles, in some large $D$-dimensional domain $\Omega$, interacting through a rotationally symmetric two-body potential $V$. The grand partition function is

$$\Xi(z) = \sum_{N=0}^{\infty} Z_N z^N$$

where

$$Z_N = \frac{1}{N!} \int_{\Omega} \prod_{j=1}^{N} d^D r_j e^{-\sum_{j<k} V(r_{jk})}$$

where $r_{jk}^2 \equiv (r_j - r_k)^2$ and we work in units where $kT = 1$. As the volume $|\Omega|$ of the domain goes to infinity, we expect, if $V$ is sufficiently short-ranged, that $\Xi(z) \sim e^{p(z)|\Omega|}$, where $p(z)$ is the pressure.

For our purposes it is easier to consider the mean density $n = z(d/dz)p(z)$, which has the expansion

$$n(z) = (\Xi(z))^{-1} \sum_{N=1}^{\infty} \frac{z^N}{(N-1)!} \int_{\Omega} \prod_{j=1}^{N} d^D r_j \delta^D(r_1)e^{-\sum_{j<k} V(r_{jk})}$$

The cluster expansion follows by writing $e^{-V(r^2)} = 1 - G(r^2)$, so that $e^{-\sum_{j<k} V(r_{jk})} = \prod_{j<k} (1 - G(r_{jk}^2))$, and expanding everything in powers of $G$. Note that if $V$ is short-ranged, so is $G$. For the density, all the contributions which diverge as $|\Omega| \to \infty$ cancel between
the numerator and denominator, and we are left with a sum of connected cluster diagrams. The coefficient of $z^N$ is a sum of connected graphs $G$, each of which has $N$ nodes labelled by $(r_1, \ldots, r_N)$, of which $r_1$ is fixed to be at the origin. The value of the graph is given by integrating over all the other $r_j$ the product of the $-G(r_{jk}^2)$ over all connected edges $(jk)$ of the graph. An example is shown in Fig. 1.

Note that the integral corresponding to a particular graph $G$ can be performed by inserting the integral representation

$$G(r_{jk}^2) = \int_0^\infty f(\alpha_{jk}) e^{-\alpha_{jk}r_{jk}^2} d\alpha_{jk}$$

on each edge: the integral over the $r_j$ then has the form

$$\int \prod_j d^D r_j \delta^{(D)}(r_1) e^{-\frac{1}{2} \sum_{jk} \sum_{\mu=1}^D r_\mu^A G_{jk} r_\mu^B}$$

where $A_G$ is a matrix which depends on the $\{\alpha_{jk}\}$, and on $G$, but not on $D$. The only feature of this we shall need is that it factorises into $D$ integrals over each component of the vectors $r_j$, and by the rules of gaussian integration it has the form

$$(2\pi)^{(N-1)D/2} (\det A_G)^{-D/2}$$

**Grassmann integration**

Grassmann coordinates behave in many ways like ordinary coordinates, except that they anticommute. For our purposes, we are going to consider a space with just two Grassmann coordinates, called $\theta$ and $\bar{\theta}$. It is useful to consider these as analogues of complex commuting coordinates $z = x + iy$ and $\bar{z} = x - iy$, so that the squared distance between two points $(\theta_j, \bar{\theta}_j)$ and $(\theta_k, \bar{\theta}_k)$ is the analogue of $(\bar{z}_j - z_k)(z_j - z_k)$, that is $|\theta_{jk}|^2 \equiv (\bar{\theta}_j - \bar{\theta}_k)(\theta_j - \theta_k)$.

We need the notion of integrating some function $f(\bar{\theta}, \theta)$ over all space. Note that because $\bar{\theta}^2 = \theta^2 = 0$, the Taylor expansion of any such function terminates: $f = A + B\theta + C\bar{\theta} + D\theta^2$. One important property of integration over all space is that we may shift the integration variables by constants, so that
\[
\int d\bar{\theta} d\theta f(\bar{\theta} + \pi, \theta + \kappa) = \int d\bar{\theta} d\theta f(\bar{\theta}, \theta)
\]  
(7)

This can only be satisfied if

\[
\int d\bar{\theta} d\theta 1 = \int d\bar{\theta} d\theta = \int d\bar{\theta} d\theta = 0
\]  
(8)

but we are allowed to choose the normalisation of

\[
\int d\bar{\theta} d\theta = \Lambda
\]  
(9)

We also have to learn about gaussian Grassmann integration: the main rule is that if \( B \) is an \( N \times N \) matrix

\[
\int \prod_j d\bar{\theta}_j d\theta_j e^{-\frac{1}{2} \sum_{jk} B_{jk} \bar{\theta}_j \theta_k} = (-\frac{1}{2} \Lambda)^N \det B
\]  
(10)

We shall not try to prove this in general [11]. Let us note the simplest case \( N = 1 \):

\[
\int d\bar{\theta} d\theta e^{-\frac{1}{2} B \bar{\theta} \theta} = \int d\bar{\theta} d\theta (1 - \frac{1}{2} B \bar{\theta} \theta) = -\frac{1}{2} \Lambda B
\]  
(11)

**Supersymmetric classical gas**

We are now going to consider a classical gas which lives in superspace. Points in superspace are labelled by \( d \) commuting coordinates, the components of \( r \), and two anticommuting coordinates \( (\bar{\theta}, \theta) \) as above. The squared distance in superspace is

\[
R_{jk}^2 = r_{jk}^2 + |\theta_{jk}|^2 = (r_j - r_k)^2 + (\bar{\theta}_j - \bar{\theta}_k)(\theta_j - \theta_k)
\]  
(12)

and our classical gas with \( N \) particles has grand partition function \( \Xi_{SS}(z) = \sum_{N=0}^\infty Z_{N,SS} z^N \), where

\[
Z_{N,SS} = \frac{1}{N!} \int \prod_j d^d r_j d\bar{\theta}_j d\theta_j e^{-\sum_{j<k} V(R_{jk}^2)}
\]  
(13)

Here ‘SS’ stands for superspace, or for ‘supersymmetric’. The supersymmetry in this case is under super-rotations in the \( d + 2 \) dimensional space. Note that there are many different
kinds of supersymmetry. This one is rather different from that considered in most particle physics applications, and is also different from the global supersymmetries used in quenched random systems in condensed matter physics.

Actually, the partition function for this supersymmetric gas is not very interesting: since $V$ depends only on the relative coordinates, the integration over the center-of-mass anticommuting coordinates gives zero, by virtue of (8), except when $N = 0$. Thus $\Xi_{SS}(z) = 1$. This is typical of supersymmetric theories. However, the mean density is non-trivial:

$$n_{SS}(z) = \sum_{N=1}^{\infty} \frac{z^N}{(N-1)!} \int \prod_{j=1}^{N} d^d r_j d \theta_j d \theta_j \delta^{(d)}(r_1) \delta(\bar{\theta}_1) \delta(\theta_1) e^{-\sum_{j<k} V(R_{jk})}$$

where $\delta(\bar{\theta}_1) \delta(\theta_1)$ just sets $\bar{\theta}_1 = \theta_1 = 0$ in whatever expression follows it.

We are going to evaluate (14) in two different ways: first we shall show that it is identical to the same expression (3) for the ordinary classical gas in $D = d - 2$ dimensions, then we shall evaluate it in another way to show that it gives branched polymers in $d$ dimensions.

**Dimensional reduction**

As before, we can make a cluster expansion of (14) by writing $e^{-V} = 1 - G$. In this case, only connected graphs survive because any disconnected pieces vanish on integrating over their center-of-mass anticommuting coordinates. Using the integral representation (4), a given graph $\mathcal{G}$ now contributes (before doing the $\{\alpha_{jk}\}$ integrations)

$$\int \prod_{j} d^d r_j d \theta_j e^{-\frac{1}{2} \sum_{j<k} (r_j A_{jk} r_k + \bar{\theta}_j A_{jk} \theta_k)}$$

$$= (2\pi)^{(N-1)d/2} (\det A_\mathcal{G})^{-d/2} (-\frac{1}{2} \Lambda)^{N-1} (\det A_\mathcal{G})$$

and we see that this is identical to (6), with $D = d - 2$, as long as we take

$$\Lambda = -1/\pi \quad .$$

To summarise, we have

$$n_{SS}^{(d,2)}(z) = n^{(d-2)}(z)$$

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In a sense, the integrations over the anticommuting coordinates cancel those over two of the commuting ones. This again is a result typical of supersymmetry.

**Branched polymers**

We haven’t yet tried to define the term ‘branched polymer.’ Physically, it is an object consisting of nodes connected by linear segments which are chains of monomers. Each node has a specific degree, the number of segments meeting at that node, which can either be 1 or any integer \( \geq 3 \). By extending this definition, we can think of the connections between each pair of monomers on the linear sections as nodes of degree 2. Each branched polymer has the topology of a tree, with a fixed number of monomers on each segment or branch. It is embedded in \( d \)-dimensional euclidean space, and there are assumed to be interactions between the monomers which (a) represent the fact that neighbouring monomers on the tree are confined within a typical radius \( a \), and (b) there is a steric repulsion between all other pairs of monomers which inhibits them from approaching each other closer than a distance also \( O(a) \). The branched polymer is assumed to be in thermal equilibrium with some kind of solvent, so that is explores its allowed phase space, each configuration occurring with a probability \( \propto e^{-\text{energy}/kT} \), according to the laws of statistical mechanics.

For enumeration purposes it is often simpler to allow the nodes to lie at the sites of a regular lattice, of spacing \( O(a) \), and then to count, with equal weights, all allowed configurations of a given total number of monomers \( N \). Depending on whether loops are allowed or not, this gives the problems of lattice animals or lattice trees. Numerical work suggests that, for large \( N \), these models exhibit a kind of critical behaviour, characterised by universal critical exponents. For example, the typical size \( R \sim N^\nu \), and the number of distinct animals (or trees) which are not related by lattice translations \( p_N \), goes like \( N^{\alpha-1} \mu^N \), where \( \mu \) is model-dependent, but the exponents \( \nu \) and \( \alpha \) are universal. The connection with conventional critical phenomena is made more clear by considering the generating function \( g(z) = \sum_N p_N z^N \), where \( z = z_c = \mu^{-1} \) is the critical point, near which \( g(z) \sim (z_c - z)^{1-\alpha} \).
We shall show that the supersymmetric model (13) gives rise to a continuum model of branched polymers which in fact is much more realistic than these lattice models.

From the supersymmetric gas to branched polymers

Go back to the formula (14) for the density and write, for each pair of particles

\[ e^{-V(R_{jk}^2)} \equiv P(R_{jk}^2) = P(r_{jk}^2 + \theta_j \theta_k) = P(r_{jk}^2) + Q(r_{jk}^2) \theta_j \theta_k \]  

(17)

where \( Q(r^2) = P'(r^2) \). So we can write the integrand in (14) as a sum of \( 2^{N(N-1)/2} \) terms, in each of which for each pair \((jk)\) we choose either the term \( P(r_{jk}^2) \) or the term \( Q(r_{jk}^2) \theta_j \theta_k \).

Each term can be represented by a subgraph of the complete graph connecting all \( N \) nodes, in which the edge \((jk)\) is included if we choose the second term, and excluded if we choose the first. See Fig. 2. Now consider performing the integrations over the anticommuting coordinates. For the same reason as above, any subgraphs not connected to the origin will vanish. Moreover, it is fairly easy to see that any subgraphs containing cycles will also vanish, since this will always involve some anticommuting coordinate being raised to a power \( \geq 2 \).

Thus the surviving subgraphs all have the form of rooted connected trees, rooted because the particle at the origin is privileged. It also easy to see that all possible such rooted trees with \( N \) nodes occur (we allow nodes of degree 2.) For a given tree \( T \) we may then relabel the coordinates, starting from the origin. The integration over the anticommuting coordinates on one particular edge of the subgraph is then of the form

\[ \int d\theta d\bar{\theta} Q(r^2) \theta \bar{\theta} = (-1/\pi)Q(r^2) \]  

(18)

We therefore have

\[ n_{SS}^{(d,2)}(z) = \sum_{N=1}^{\infty} z(-z/\pi)^{N-1} N \mathcal{Z}_N \]  

(19)

where

\[ \mathcal{Z}_N = \sum_{T} \frac{1}{N!} \int \prod_{j=1}^{N} d^d r_j \delta^{(d)}(r_1) \prod_{j \sim k} Q(r_{jk}^2) \prod_{j \not\sim k} P(r_{jk}^2) \]  

(20)
where \( j \sim k \) means that \( j \) and \( k \) are neighbouring nodes on \( T \), and \( j \not\sim k \) means that they are not, and each pair \((jk)\) is counted just once.

For suitable potentials \( V \), we can interpret this as the partition function for a branched polymer with the topology of \( T \): \( P(r_{jk}^2) = e^{-V(r_{jk}^2)} \) is the Boltzmann weight representing the steric repulsion between different parts of the polymer, and \( Q(r_{jk}^2) \) then is the weight for two neighbouring nodes, which keeps them a distance \( O(a) \) apart. If the original gas had only repulsive forces, then \( P' \geq 0 \) so that \( Q \) may be interpreted as a weight. Fortunately it is quite easy to arrange a suitable function \( Q \). An example is illustrated in Fig. 2. A special case is to choose \( V(r^2) = V_0 > 0 \) for \( r < a \) and zero for \( r > a \). Then

\[
Q(r^2) = (1 - e^{-V_0})\delta(r^2 - a^2) = (2a)^{-1}(1 - e^{-V_0})\delta(r - a) \tag{21}
\]

If we take the limit \( V_0 \to \infty \) corresponding to a hard core repulsive gas, then the corresponding branched polymer model may be thought of as consisting of balls of radius \( a/2 \), with neighbouring balls on the tree constrained to touch each other, and a hard core repulsion between the rest.

Putting together all the pieces of the puzzle, we have shown that the generating function for rooted branched polymers in \( d \) dimensions is related to the mean density of a repulsive gas in \( D = d - 2 \) dimensions by

\[
z(d/dz)p(z) = n(z) = \sum_{N=1}^{\infty} z(-z/\pi)^{N-1}N Z_N \tag{22}
\]

By integrating with respect to \( z \), we finally end up with a relation between the pressure \( p(z) = \lim_{|\Omega|\to\infty} |\Omega|^{-1} \ln \Xi(z) \) of the classical gas and the generating function for branched polymers:

\[
p(z) = -\pi \Sigma(-z/\pi) \tag{23}
\]

where

\[
\Sigma(\zeta) = \sum_{N=1}^{\infty} Z_N \zeta^N \tag{24}
\]

Eq. (23) is the main result of Brydges and Imbrie [10].
Simple examples

1. $d = 2$

When $d = 2$ then $D = 0$, a particularly simple case, since the universe is all at one point [12]. In that case, we do not have to worry about the thermodynamic limit either. The grand partition function is

$$\Xi(z) = \sum_{N=0}^{\infty} \frac{z^N}{N!} e^{-\frac{1}{2}N(N-1)V(0)}$$

(25)

since if there are $N$ particles at the origin, there are $\frac{1}{2}N(N-1)$ interactions between them. The remarkable thing is that the branched polymer partition function in $d = 2$ depends only on the value of the potential $V(r)$ at the origin. This is a consequence of the supersymmetry. If we relax the condition that $Q = P'$, it is no longer true. Although the sum in (25) cannot be performed explicitly, it can be shown [13] that all the zeroes of $\Xi(z)$ lie on the negative real axis and are distinct. The closest one to the origin determines the large $N$ behaviour of $\mathcal{Z}_N$, and gives rise to the branched polymer singularity.

In the hard core case $V(0) \to \infty$, there can only be zero or one particles, so we have

$$p(z) = \ln \Xi(z) = \ln(1 + z)$$

(26)

so that

$$\Sigma(\zeta) = -(1/\pi) \ln(1 - \pi\zeta)$$

(27)

and

$$\mathcal{Z}_N = \pi^{N-1}/N$$

(28)

It is very hard to see how this simple result could be obtained by other means. While it is easy to check for $N = 1, 2$, it already is non-trivial for $N = 3$, while for $N \geq 4$ the integrals involved in each contribution appear intractable.

\[1^1\]Note that this differs by factors of 2 from equation (1.4) of Ref. [10], since we have $Q = (1/2a)\delta(r-a)$, while these authors take $Q = \delta(r-1)$. 

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2. $d = 3$

Interacting classical gases in $D = 1$ can generally be solved exactly only in the case of infinite hard core repulsion, i.e. $V(0) \to \infty$. We can think of the particles as rods which we now take to be of unit length. If the total length is $L$ and we use periodic boundary conditions then since configuration space has volume $(L-N)^N$, we have $Z_N = (L-N)^N/N!$

A classic calculation in statistical mechanics then shows that, in the thermodynamic limit, $p = n/(1-n)$ (the denominator is just the excluded volume term in van der Waals' equation!) and so $z(dp/dz) = n = p/(1+p)$. Integrating, we find an implicit equation for the pressure: $z = pe^p$. Remarkably, this can be solved explicitly to find $p(z)$ as a power series in $z$ [Hint: write $\oint p(z)dz/z^{N+1}$ as a similar contour integral over $p$]

$$p(z) = \sum_{N=1}^{\infty} \frac{(-N)^{N-1}}{N!} z^N$$

(29)

Using the main result (23) this expansion once again gives some highly nontrivial identities for sums of certain integrals in three dimensions.

If we are just interested in the branched polymer singularity note that $z(p)$ has a quadratic turning point at $z = -z_c = -e^{-1}$, so that $p(z)$ has a square root branch point there. This is consistent with the square root singularity at the Yang-Lee edge in $D = 1$: it is an easy exercise in transfer matrices to solve the nearest neighbour Ising model in a purely imaginary field, and check this.

**Correlation functions**

Although we have so far considered only the mean density, it is easy to generalise dimensional reduction to correlation functions:

$$\langle n(r_1)n(r_2)\ldots n(r_p)\rangle_{SS}^{(d,2)} = \langle n(r_1)n(r_2)\ldots n(r_p)\rangle_{d-2,0}$$

(30)

which is valid as long as the points $r_j$ all lie in the $(d - 2)$-dimensional subspace, and at $\theta = \bar{\theta} = 0$. If we now expand the left hand side in terms of $P$ and $Q$ as before we find the
generating function for \( p \) different branched polymers, which exclude each other’s volume, rooted at the points \( r_j \). Of course, for \( p = 2 \) the correlation function enjoys full rotational invariance in superspace. Thus if we take \( r_1 = 0 \) and \( r_2 = r \) and we let

\[
G(r^2 + \theta \theta) = \langle n(0)n(r, \theta, \theta) \rangle_{SS}^{(d,2)}
\]  

(31)

then \( G(r^2) \) is the same as the density-density correlation function for the \( D \)-dimensional ordinary gas. Writing now \( G(r^2 + \theta \theta) = G(r^2) + \theta \theta G'(r^2) \), the first term gives as before the correlation function between two different but interacting branched polymers, while, after performing the Grassmann integrations, we see that the second term, proportional to \( G'(r^2) \), gives the density-density correlation function on a single branched polymer. That these are related is a consequence of the supersymmetry. This is all spelled out in full detail in Ref. [14]. Since for \( D = 1 \) the correlation function can in some cases be computed exactly, so can that for branched polymers in \( d = 3 \). This was first carried out using the Yang-Lee model by Miller [15].

**Extensions**

I’ll finish by briefly mentioning some simple extensions of this beautiful idea.

**Bells and whistles**

It is possible to decorate the nodes of the branched polymer with other degrees of freedom, e.g. to consider that they are of different species, and that the interactions between them can depend on this. Under dimensional reduction this becomes a classical gas with the same decoration. It would be nice if we could find a different universality class at negative fugacity. Field theory suggests that this might be the case (we choose a theory with a \( \phi^{2n+1} \) interaction with \( n \geq 2 \), rather than \( \phi^3 \)) but investigations so far [13] have shown that, at least for physical values of the interactions, the universal repulsive singularity is robust, and nothing new happens.
Adsorption at a wall or a plane

The astute listener should have realised that all that is needed for dimensional reduction to work is rotational invariance in two of the \( d \) commuting dimensions, so that integrals over these can be cancelled by the two anticommuting ones. Thus, for \( d \geq 3 \), we are free to add an arbitrary potential \( U(x) \) depending on one of the commuting coordinates \( x \). The generating function for branched polymers in \( d \) dimensions in the presence of this potential will be related to that of a \( D \)-dimensional classical gas in the same manner as (23).

As an example take a hard core repulsive gas in \( d = 3 \), and \( e^{-U(x)} = 1 + \lambda \delta(x) \), with \( \lambda > 0 \), representing an attractive potential on the plane \( x = 0 \). A simple calculation then shows that the density at a point close to, but not on, the plane is modified to

\[
n(z) = \frac{n_0(z)}{1 + \lambda n_0(z)}
\]

(32)

where \( n_0(z) \) is the unperturbed density. For small \( \lambda \), the nearest singularity to the origin is still at \( z = -z_c \), the \( d = 3 \) branched polymer singularity. However, for large enough \( \lambda \) it instead comes from the vanishing of the denominator, and is generally a simple pole. This is the branched polymer singularity in \( d = 2 \). In this regime the branched polymer is confined to the vicinity of the plane \( x = 0 \). The critical value of \( \lambda \) at which these singularities collide represents an adsorption transition. The case of an attractive wall, where \( U = +\infty \) for \( x < 0 \), may be dealt with similarly.

Dimensional reduction by 4

We do not have to stop at adding one pair of anticommuting coordinates - for example we can add another, in which case we get dimensional reduction by four. But the equivalent \( d \)-dimensional model then involves interaction weights proportional to \( P, P' \) and \( P'' \), and the last typically has to change sign, so is unphysical.
Intriguing Connections

If one takes the scaling limit of the expression (25) as $z \to -z_c$ and $V(0) \to 0$, one finds something proportional to an Airy function. The Airy integral arises in many contexts, but another was recently found by Richard, Guttmann and Jensen [16], who argued that it occurs in the scaling limit of self-avoiding closed loops in the plane, weighted by their length and the area enclosed within. The parameters conjugate to these are the fugacity, analogous to $z$, and the (negative) pressure inside the loop, which plays the same role as $V(0)$ in the branched polymer generating function. Using the field theory arguments of Parisi and Sourlas [5], I have argued elsewhere [17] why an Airy function is expected in this context. This also explains why the fractal dimension of self-avoiding loops is $\frac{4}{3}$. But it would be very interesting to provide a direct connection between branched polymers and self-avoiding loops – is perhaps the external perimeter (suitably defined) of a large branched polymer in the same universality class as that of ordinary self-avoiding loops?

Acknowledgements

Over the years I have benefited from conversations on this subject with N. Sourlas, Y. Shapir and J. Miller, and more recently with J. Imbrie and R. Rajesh. These lectures were written while the author was a member of the Institute for Advanced Study. He thanks the School of Mathematics and the School of Natural Sciences for their hospitality. This stay was supported by the Bell Fund, the James D. Wolfensohn Fund, and a grant in aid from the Funds for Natural Sciences.

Since these notes were written, another review by Imbrie [18] has appeared.
REFERENCES


FIG. 1. A typical cluster diagram for the mean density.

FIG. 2. A branched polymer configuration. Solid lines correspond to the nearest neighbor interaction $Q$, and dashed lines to repulsive interactions $P$.

FIG. 3. Typical forms for $P$ (dashed curve) and $Q = P'$ (solid).