Yang-Mills Theory on Loop Space*

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Abstract

We will describe some mathematical ideas of K. T. Chen on calculus on loop spaces. They seem useful to understand non-abelian Yang–Mills theories.

1 Classical Gauge Theories

The mathematical apparatus to describe gauge theories was discovered almost simultaneously with the work of Yang and Mills: the theory of connections on principal fiber bundles[1]. We will now describe the three basic examples of classical gauge theories in this language before explaining the difficulties of formulating their quantum counterparts.

Let X be a differentiable manifold and G a compact Lie group. The most interesting case is when X , which represents space-time, is four dimensional. Also $G = SU(N)$ is the most interesting case, the value of N being three for quantum chromodynamics. By the imposition of appropriate boundary conditions, we can often restrict attention to a compact space X.

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A gauge field A is a connection on the principal fiber bundle $G \to P \to X$. It is sufficient to consider the case where P is topologically trivial, so that it is diffeomorphic to the product $X \times G$: all the physically interesting phenomena occur already in this case.

The different gauge theories are characterised by differential equations satisfied by the connection. The simplest is Chern-Simons-Witten theory which is the theory of flat connections. The curvature $F(A) = dA + A \wedge A$ is required to vanish:

$$dA + A \wedge A = 0. \tag{1}$$

The set of solutions of this equation (the classical phase space) is the same as the set of equivalence classes of representations of the fundamental group of X in G. Thus it is a finite dimensional space. The quantum version-essentially due to Witten [2] of even this simplest of all gauge theory leads to profound new results in topology: a better understanding of the Jones invariant of knots theory and the Witten invariants of three manifolds. Note that Chern-Simons-Witten theory does not use any notion of metric on X: it is a topological field theory. The most interesting case is when X is three dimensional although the defining equations make sense in all dimensions.

The next simplest gauge theory of interest[3] is the self-dual Yang–Mills Theory. In this case, X is a four dimensional manifold with a Riemannian metric. The defining partial differential equation says that half the components of curvature vanish:

$$F(A) = *F(A) \tag{2}$$

where $*$ is the Hodge dual that maps two forms to two forms. (In fact these equations depend only on the conformal class of the metric tensor of X).

The deepest gauge theory of all is Yang–Mills theory, where the connection satisfies

$$d_A * F(A) = 0 \tag{3}$$

where d_A is the covariant derivative, $d_A * F(A) = d * F(A) + A \wedge *F(A) + [*F(A)] \wedge A$. Although other cases can be studied as toy models, the most

interesting case is when X is four dimensional and the metric tensor on it is of Lorentzian signature. Even in this case, there is by now a complete understanding of the initial value problem [4]: we can regard the classical Yang–Mills theory as well-understood.

2 Wilson Loops

A connection can be thought of as a one form on X valued in the Lie algebra of G: $A \in \Lambda^1(X) \times \underline{G}$. This identification depends on a choice of trivialization. A change of trivialization is a 'gauge transformation' $g : X \to G$ which acts on the gauge field as follows:

$$A \mapsto gAg^{-1} + gdg^{-1}. \tag{4}$$

All geometrically and physically meanigful quantities must be invariant under this transformation. Even the curvature $F(A) = dA + A \wedge A$ is not invariant: it transforms in the adjoint representation:$F(A) \mapsto gF(A)g^{-1}$. The trace of $F(A)$ and its powers are gauge invariant. But they do not provide a complete set of gauge invariant quantities from which the underlying connection can be recovered; this is especially obvious if X is not simply connected.

The most natural gauge invariant quantities are the traces of the holonomy of a curve. Define a function $S : [0, 2\pi] \to G$ by the condition:

$$\frac{dS(t)}{dt} + \gamma^* A(t)S(t) = 0. \tag{5}$$

(Here $\gamma^* A$ is the pullback of A to a one-form on the interval.) Eventhough $\gamma^* A(t)$ is periodic, the solution to the above equation will not be: $U(\gamma) = S(0)^{-1}S(2\pi)$ is the parallel transport (holonomy) around the closed curve. Under a gauge transformation, $U(\gamma) \mapsto g(\gamma(0))U(\gamma)g(\gamma(0))^{-1}$ so that the trace $W(\gamma) = \frac{1}{N} \operatorname{tr} U(\gamma)$ is indeed gauge invariant. This quantity is called the 'Wilson loop' in physics jargon.

The Wilson loop is thus a complex valued function on the space of all closed curves in X. By solving the parallel transport equation in a power

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series we can get the following expansion for the Wilson loop:

$$W(\gamma) = \int_{\Delta_n} \frac{1}{N} \, \text{tr} \, [A_{\mu_n}(\gamma(t_n)) \cdots A_{\mu_1}(\gamma(t_1))] \dot{\gamma}^{\mu_n}(t_n) \cdots \dot{\gamma}^{\mu_1}(t_1) dt_n \cdots dt_1.$$
(6)

where Δ_n is the simplex $t_1 \leq t_2 \leq \cdots t_n$.

Is it possible to reformulate the above gauge theories in a new way where $W(\gamma)$ is the basic variable?

In the simplest case of Chern-Simons-Witten theory the answer is obvious at least at the classical level. The condition of flatness of the connections translates to the requirement that $W(\gamma)$ be invariant under continuous deformations of the loop: that the Wilson loop be a function $W : \Omega X \to C$ whose derivative is zero:

$$dW(\gamma) = 0.$$
(7)

In the quantum theory there will be singularities whenever γ intersects itself: to get a sensible answer, γ must be an embedding of the circle into X. Then $W(\gamma)$ would be unchanged under deformations of this embedding; i.e., a knot invariant. In fact we can calculate the amount by which $W(\gamma)$ changes as γ is deformed through self-intersections, giving some difference equations. The resulting difference equation for $W(\gamma)$ is related to Vasiliev's approach to knot invariants.

In the case of self-dual Yang–Mills theory, the curvature is 'half-flat'. This ought to translate to a condition that the Wilson loop is an analytic function on the space of loops:

$$\bar{\partial}W(\gamma) = 0.$$
(8)

The question arises: is there a kind of calculus on loop space with respect to which these equations make sense? What is then the way to rewrite classical Yang–Mills theory this way?

Only after we understand these questions we can hope to formulate and solve quantum Yang–Mills theories this way.

3 Quantum Gauge Theories

There are several ways of passing from a classical theory to quantum theory. All of them fail to provide a mathematically well-defined quantum theory in the case of four dimensional Yang–Mills theories due to divergences that are characteristic of quantum field theories. There is every reason to believe that such a theory exists however: the other two classes of gauge theories as well as Yang–Mills theories in space-time dimension less than four are free of divergences. More importantly, the profound work of 't Hooft [5] shows that these divergences can be removed to all orders in perturbation theory even in four dimensions: quantum Yang–Mills theory is renormalizable. Nevertheless these difficulties are formidable.

Loosely speaking, in the quantum field theory, the fields are random variables. The probability of a particular configuration is proportional to $\epsilon - S(A)$ where $S(A)$ is the action. (The stationary points of $S(A)$ are the solutions of the classical field equations.) For Yang-Mills theory, for example, $S(A) = \frac{1}{2\alpha} \int_X \text{ tr } F * F$.

All physical quantities follow from the expectation values of the fields, ('correlation functions' or Green's functions) such as

$$G_{\mu_1 \cdots \mu_n}(x_1, \cdots x_n) = < \frac{1}{N} \text{ tr } A_{\mu_1}(x_1) \cdots A_{\mu_n}(x_n) > . \tag{9}$$

We now recognize that the expectation value of the Wilson loop is a generating function for all these Green's functions:

$$< W(\gamma) > = \int_{\Delta_n} G_{\mu_1 \cdots \mu_n}(\gamma(t_1), \cdots \gamma(t_n)) \dot{\gamma}^{\mu_n}(t_n) \cdots \dot{\gamma}^{\mu_1}(t_1) dt_n \cdots dt_1 \tag{10}$$

This also projects out the gauge invariant part of the correlation functions.

The first main problem in the field (apart from the construction of quantum Yang-Mills theory) is that this expectation value has the asymptotic form (Wilson's Area law)

$$W(\gamma) \sim e^{-T\text{Area}(\gamma)} \tag{11}$$

for large loops. This is has been known for some time in two space-time dimensions: an easy result. More recently in it has been shown (at a level

of rigor common in theoretical physics) in three space-time dimensions-
a profound physical result. It remains open even at this level in four
space-time dimensions.

It would be very helpful to rewrite gauge theories as field theories on
the loop space of space-time. How does one write differential equations in
this infinite dimensional space? Surprisingly, much of the mathematical
apparatus needed for this has already developed in the work of K. T.
Chen in topology [6]. In modern language, K. T. Chen developed certain
classical topological field theories on loop spaces.

4   K. T. Chen's Iterated Integrals

The set of loops on space-time is an infnite dimensional space; calculus on
such spaces is in its infancy. It is too early to have rigorous definitions of
continuity and differentiablity of such functions. Indeed most of the work
in that direction is of no value in actually solving problems of interest
(rather than in showing that the solution exists.)

K.T. Chen's idea is to think of a function on loop space as a sequence
of functions finite dimensional spaces. The simplest example of a function
on loop space is the integral of a one-form on X around a curve:

$$\int_\gamma \omega = \int_0^{2\pi} \omega_\mu(\gamma(t))\dot\gamma^\mu(t)dt. \tag{12}$$

More generally, we can have functions arising from multiple integrals like

$$\int_{0 \le t_1 \le t_2} \omega_{\mu_2\mu_1}(\gamma(t_2),\gamma(t_1))\dot\gamma^{\mu_2}(t_2)\dot\gamma^{\mu_1}(t_1)dt_2 dt_1. \tag{13}$$

The integrand is a tensor field $\omega_{\mu_1\mu_2}(x_1,x_2)$ on $X \times X$ which does not need
to be symmetric: it is an element of the tensor product $\Lambda^1(X) \otimes_C \Lambda^1(X)$.
We can more generally imagine a function on loop space as a formal power
series in such integrals:

$$\int_\gamma \omega := \int_{\Delta_n} \omega_{\mu_1\cdots\mu_n}(\gamma(t_1),\cdots\gamma(t_n))\dot\gamma^{\mu_n}(t_n)\cdots\dot\gamma^{\mu_1}(t_1)dt_n\cdots dt_1. \tag{14}$$

The coefficient of the nth term is an element of the n-fold product $\mathcal{T}^n = \Lambda^1(X)\otimes_C\cdots\Lambda^1(X)$. Thus the sequence of tensor fields $\omega\mu_1\cdots\mu_n(x_1,\cdots x_n)$

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can be thought of as defining a function on the loop space LX. There is a technical complication: there are certain tensor field which give zero upon integration a closed curve. (For example, $\omega = df$, an exact one-form on X.) Chen has identified this subspace $\mathcal{K}$. Thus we can identify the algebra of functions on LX as the quotient space $\mathcal{T}/\mathcal{K}$ where $\mathcal{T} = \sum_{n=0} \mathcal{T}^n$.

This is exactly the class of functions that we need to understand Yang–Mills theory. Notice that this fits exactly with the expansion of the Wilson loop expectation value: the Green's functions of Yang–Mills theory are the coefficients. Moreover the kernel K is exactly the change in the Green's functions due to a gauge transformation.

There is an important change to Chen's work that is needed to apply it to the study of quantum gauge theories. The Green's functions are singular when a pair of points coincide (this is true even in free field theories with linear field equatons). Thus the curves we allow should not interesect themselves: they should be embeddings of the circle in space-time. A function on the space of embeddings EX can be expanded as above in iterated integrals, but the coefficients are tensor fields on the 'configuration space' of X,

$$F(X,n) = \{(x_1, \cdots x_n)|x_i \neq x_j \text{ for } i \neq j\}. \tag{15}$$

These configuration spaces are interesting objects in themselves in topology.

5 Formal Power Series in One Variable

It is useful to look back in history to a time when calculus of one variable itself was new to see how we should develop a calculus of an infinite number of variables. Having understood how to differentiate and integrate polynomials, functions were thought of as infinite series on which similar operations couldbe defined. Even without a theory of convergence of series (which was developed later) it is possible to do this in a completely rigorous way: this is the theory of formal power series[7].

We define a formal power series $a = (a_0, a_2, \cdots)$ to be a sequence of complex numbers, which do not need to decrease. Define the sum product

and differential of such sequences as follows:

$$[a+b]_n = a_n + b_n, \quad [ab]_n = \sum_{p+q=n} a_p b_q, \quad [da]_n = (n+1)a_{n+1}. \quad (16)$$

This can be easily verifed to be a commutative algebra on which d is a derivation:

$$d(ab) = (da)b + adb. \quad (17)$$

If all but a finite number of entries are zero, such a formal power series defines a polynomial $a(z) = \sum_{n=0} a_n z^n$ and the above rules are the correct rules of adding multiplying and differentiating polynomials. We simply note that these rules make sense even on infinite sequences even without any convergence conditions on them.

We can derive similar rules for adding and multiplying functions on loop spaces: a calculus on an infinit dimensional space can be developed first as one on a sequence of functions each depending only on a finite number of variables.

6   Calculus on Loop Space

We define a formal function on the space of embeddings EX as a sequence of tensors

$$\omega = (\omega_0, \omega_1, \omega_2 \cdots) \quad (18)$$

where ω_n is a covariant tensor on the configuration space $F(X, n)$; we require these tensor fields to be of rank one in each variable:

$$\omega_n = \omega_{\mu_1, \mu_2, \cdots \mu_n}(x_1, x_2, \cdots x_n) dx_1^{\mu_1} \otimes dx_2^{\mu_2} \cdots \otimes dx_m^{\mu_n}. \quad (19)$$

We allow these tensor fields to have singularities as $x_i \to x_j$, since they only need to be well-defined on the configuration space.

It will be useful to combine the discrete label μ and the continuous label x into a single one i and to think of such a tensor field as ω_I where $I = ((\mu_1, x_1), (\mu_2, x_2), \cdots (\mu_n, x_n)$ is a sequence of discrete and continuos labels.

By thinking of a function as a series of iterated integrals as above (which would make perfect sense if only a finite number of these tensors are

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non-zero) we can derive rules for addition (the obvious pointwise addition will do) and multiplication:

$$[\omega \circ \phi]_I = \sum_{S \subset \{1,2,\cdots r\}} \omega_{I_S} \phi_{I_{\bar{S}}}. \tag{20}$$

This is called the shuffle product: the sum is over ways of subdividing the sequence I into a subsequence with labels in the subset S and the complimentary sequence labelled by $\bar{S}$. This multiplication is obviously commutative and can be verified to be associative; we call this the shuffle algebra $Sh(X)$. (This shuffle product is in fact the dual of the co-multiplication in the usual tensor algebra.)

There is another,non-commutative,multiplication as well defined on these sequences of tensors: the concatenation:

$$[\omega * \phi]_{\mu_1,\mu_2,\cdots n}(x_1, x_2, \cdots x_n) = \sum_{r=0}^{n} \omega_{\mu_1,\mu_2,\cdots \mu_r}(x_1, x_2, \cdots x_r) \tag{21}$$
$$\phi_{\mu_{r+1},\cdots \mu_n}(x_{r+1}, \cdots x_n).$$

This corresponds to the multiplication of loops (with a common base point) where one follows along the first loop and then the other. The two multiplication fit into a bialgebra, indeed even a Hopf algebra. These operations of tensor fields on configuration spaces as well as the idea that sequences of configuration spaces can be viewed as approximations to spaces of embeddings are ideas originating in algebraic topology.

The set of tensors that give zero upon integration on a curve for an ideal $\mathcal{K}$ generated by elements of the form

$$u * df * v \tag{22}$$

where $f \in \Lambda^0(X)$. The shuffle algebra $Sh(X)/\mathcal{K}$ may be thought of as a model for the algebra of functions on the infinite dimensional space EX.

To define a differentiation, we first define the shift operator $\hat{\alpha}_\mu(x)$,

$$[\hat{\alpha}_\mu(x)\omega]_{\mu_1 \cdots \mu_n}(x_1, \cdots x_n) = \omega_{\mu\mu_1 \cdots \mu_n}(x, x_1, \cdots x_n). \tag{23}$$

Then the exterior derivative is the operator $\partial_\mu \hat{\alpha}_\nu + \hat{\alpha}_\mu \hat{\alpha}_\nu - \mu \leftrightarrow \nu$ applied to the tensor:

$$[d\omega]_{\mu\nu\mu_1 \cdots \mu_n}(x, x_1, \cdots x_n) = \partial_\mu \omega_{\nu\mu_1 \cdots \mu_n}(x, x_1, \cdots x_n) + \tag{24}$$
$$\omega_{\mu\nu\mu_1 \cdots \mu_n}(x, x, x_1, \cdots x_n) - \mu \leftrightarrow \nu$$

This operation is derivation with respect to the shuffle multiplication given above. Also, (with appropriate generalization to exterior derivative of higher order forms) it satisfies $d^2 = 0$. Chen uses the corresponding de Rham cohomology to derive results in homotopy theory.

We can now check that the condition $dW = 0$ on the Wilson loop follows from the flatness of the connnection; it is the classical equation of motion of Chern-Simons–Witten theory translated into loop space. Classical Yang–Mills equations also become a linear equation on the Wilson loop $YW = 0$ where the differential operator Y is

$$Y_\nu = \partial^\mu(\partial_\mu\hat{\alpha}_\nu - \partial_\nu\hat{\alpha}_\mu + [\hat{\alpha}_\mu, \hat{\alpha}_\nu]) + \hat{\alpha}^\mu(\partial_\mu\hat{\alpha}_\nu - \partial_\nu\hat{\alpha}_\mu + [\hat{\alpha}_\mu, \hat{\alpha}_\nu]). \quad (25)$$

Because of the singularities in the correlation functions, these equations will become singular in the quantum theory. It remains a challenge to show that these singularities can be removed: to show that the loop equations are renormalizable, beyond perturbation theory.

The Migdal-Makeenko equations [8] of the large N limit of gauge theories can now be written also as differential equations on loop space. The framwork of Chen seems better suited to resolving singularities in them than the original ideas of Migdal and Makeenko on Stokes functions in removing singularities as well as solving these equations. We hope to return to these issues elsewhere in more detail [8].

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