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Matrix Regularization of Symplectic and Conformally Invariant Theories

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Branes Workshop at Argonne National Laboratories, Oct 2003



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Symplectic and Conformal Transformations

A metric g_{ij} on a 2-manifold Σ defines:

- (i) an area element (invertible 2-form) $\omega_{ij} = \sqrt{g}\epsilon_{ij}$
- (ii) and a complex structure $J_j^i = \frac{1}{\sqrt{g}}g^{ik}\epsilon_{kj}$.

These contain complementary pieces of information about the metric, the first invariant under area preserving (symplectic) diffeomorphisms while the second is invariant under conformal transformations.



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Symplectic Field Theories

There are thus two classes of field theories whose continuum limits are invariant under these transformations.

An example of a symplectic field theory is two dimensional Yang–Mills theory, $S = \int_{\Sigma} \text{tr} F(A) * F(A)$. This is exactly solvable.

A more intricate example is the classical field theory of incompressible fluid flow in two dimensions, since it can be viewed as the geodesic equations on the group of area preserving diffeomorphisms (Arnold).

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p, \quad \nabla \cdot u = 0. \quad (1)$$



It is best to eliminate pressure by taking the curl of this equation, writing it in terms of vorticity $\omega = \nabla \times u$:

$$\frac{\partial \omega}{\partial t} = \partial_a \omega \epsilon_{ab} \partial_b \int G(x, y) \omega(y) d^2 y. \quad (2)$$

Here, $G(x, y)$ is the Green's function of the Laplace operator: vorticity determines velocity through

$$u_a(x) = \int \epsilon_{ab} \omega(y) \partial_b G(x, y) d^2 y \quad (3)$$

The analogy of this to the equations of a rigid body

$$\frac{dL}{dt} = L \times (I^{-1} L) \quad (4)$$



could not have escaped Euler. Vorticity is analogous to angular momentum; the Laplace operator is analogous to moment of inertia I .



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Hamiltonian Formalism of Euler Equations

The analogue of the angular momentum Lie algebra is the algebra of symplectic transformations on the plane:

$$[f_1, f_2] = \epsilon^{ab} \partial_a f_1 \partial_b f_2. \quad (5)$$

Every function $f : R^2 \rightarrow R$ corresponds to an observable of two dimensional hydrodynamics, $\omega_f = \int f(x) \omega(x) d^2x$. The above Lie bracket of functions gives the Poisson bracket for vorticity:

$$\{\omega(x), \omega(y)\} = \epsilon^{ab} \partial_b \omega(x) \partial_a \delta(x - y). \quad (6)$$



Euler equations follow from these if we postulate the hamiltonian to be the total energy of the fluid,

$$H = \frac{1}{2} \int u^2 d^2x = \frac{1}{2} \int G(x, y) \omega(x) \omega(y) d^2x d^2y. \quad (7)$$

The rigid body equations are the equations for a geodesic in the rotation group, with respect to the metric defined by the moment of inertia. In the same way, the Euler equations of hydrodynamics describe geodesics in the group of symplectic diffeomorphisms with respect to the anisotropic metric defined by the kinetic energy.

The quantities $Q_k = \int \omega^k(x) d^2x$ are conserved for any $k = 1, 2, \dots$: these are the Casimir invariants. In spite of the infinite number of conservation laws, two dimensional fluid flow is chaotic.





The Stochastic Navier-Stokes Equation

A chaotic system is sensitive to small changes in initial conditions or its environment. We can model this by adding a Gaussian noise to the equations. Any time we have noise, we must also have dissipation—otherwise the system will heat up to infinite energy. The dissipation of hydrodynamics is modelled well by viscosity (not necessarily the molecular viscosity).



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The Navier-Stokes equations with a random force field¹

$$\frac{Du}{Dt} = -\nabla p + \nu \Delta u + f \quad (8)$$

would model this situation nicely. Taking its curl to eliminate pressure again, we get

$$\frac{\partial \omega}{\partial t} = \partial_a \omega \epsilon_{ab} \partial_b \int G(x, y) \omega(y) d^2 y - \nu \Delta \omega + \eta. \quad (9)$$

The dissipative term can be viewed as the gradient of energy $H = \frac{1}{2}(\omega, \Delta^{-1} \omega)$, with respect to the contravariant metric tensor in the vector space of vorticities given by the operator Δ^2 .

¹Here, Δ is the Laplace operator $\Delta = -\nabla^2$. It is a positive operator.





The Fluctuation-Dissipation Theorem

The noise η is best modelled as a Gaussian with zero mean and covariance:

$$\langle \eta(t, x)\eta(t', x') \rangle = \delta(t - t')\gamma(x, x'). \quad (10)$$

The distribution $\gamma(x, x')$ should now be determined by physical considerations. The condition that the average energy pumped into the system by noise should be balanced by the dissipation is that the covariance of the fluctuations should be proportional to the dissipation tensor:

$$\gamma(x, x') = \Delta^2\delta(x, x').$$

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The Need for Regularization

Together we now can write a Fokker-Plank equation for the Navier-Stokes equation. Unfortunately, this is a hopelessly singular equation, because of the above distributions. We need to regularize the system. What is a regularization that preserves the Lie algebra structure?

The Lie algebra of symplectic transformations can be thought of as the limit of the unitary Lie algebra as the rank goes to infinity. To see this, impose periodic boundary conditions and write in terms of Fourier



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coefficients as

$$H = (L_1 L_2)^2 \sum_{m \neq (0,0)} \frac{1}{m^2} |\omega_m|^2, \quad \{\omega_m, \omega_n\} = -\frac{2\pi}{L_1 L_2} \epsilon_{ab} m_a n_b \omega_{m+n}. \quad (11)$$

Using an idea of Fairlie and Zachos, we now truncate this system by imposing a discrete periodicity mod N in the Fourier index m ; the structure constants must be modified to preserve periodicity and the Jacobi identity:

$$\{\omega_m, \omega_n\} = \frac{1}{\theta} \sin[\theta(m_1 n_2 - m_2 n_1)] \omega_{m+n \pmod N}, \quad \theta = \frac{2\pi}{N}. \quad (12)$$

This can be thought of as a 'quantum deformation' of the Lie algebra of symplectic transformations. This is the Lie algebra of $U(N)$.





Regularized hamiltonian

The hamiltonian also has a periodic truncation

$$H = \frac{1}{2} \sum_{\lambda(m) \neq 0} \frac{1}{\lambda(m)} |\omega_m|^2, \quad (13)$$

$$\lambda(m) = \left\{ \frac{N}{2\pi} \sin \left[\frac{2\pi}{N} m_1 \right] \right\}^2 + \left\{ \frac{N}{2\pi} \sin \left[\frac{2\pi}{N} m_2 \right] \right\}^2. \quad (14)$$

This hamiltonian with the above Poisson brackets describe the geodesics on $U(N)$ with respect to an anisotropic metric. We are writing it in a basis in which the metric is diagonal.



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Now we can write the regularized Navier-Stokes with random sources. We will not have the problem of 'closure' that plagues many approaches to turbulence because our regularization preserves the symmetries (the Lie algebra structure).

$$\frac{d\omega^m}{dt} = r^{mn} \partial_p H - D^{mn} \partial_n H + \eta^n \quad (15)$$

where the Poisson tensor is given in terms of the structure constants of the Lie algebra $r^{mn} = c_p^{mn} \omega_p$. The dissipation tensor is diagonal in our basis and has the square of the eigenvalues of the discrete Laplacian:

$D^{mn} = \nu \delta^{mn} \lambda(m)^2$. Also,

$$\langle \eta^m \eta^n \rangle = Q_{mn} \quad (16)$$

for some positive covariance matrix Q .





The Regularized Fokker-Plank Equation

The Fokker-Plank equation is now

$$\frac{\partial W}{\partial t} = \partial_m (r^{mn} \partial_m H W + D^{mn} \partial_n H W + Q^{mn} \partial_m W). \quad (17)$$

To have an equilibrium solution of the form $W \sim e^{-\beta H}$, we need to postulate a relation between the dissipation and fluctuation tensors:

$D^{mn} = \beta Q^{mn}$. It is not easy to show that the limit $N \rightarrow \infty$ exists: about as hard as constructing a quantum field theory. Assuming the limit exists we can make some predictions about two dimensional turbulence.





Entropy of Incompressible Flow

The constants $Q_m = \int \omega^m(x) d^2x$ define some infinite dimensional surface in the space of all functions on the plane. The micro-canonical entropy (a la Boltzmann) of this system will be the log of the volume of this surface. How to define this volume of an infinite dimensional manifold? We can compute it in the regularization and take the limit. It is convenient to regard ω as a matrix by defining $\hat{\omega} = \sum_m \omega_m U(m)$ where $U(m) = U_1^{m_1} U_2^{m_2}$ with the defining relations $U_1 U_2 = e^{\frac{2\pi i}{N}} U_2 U_1$. In this basis, the Lie bracket of vorticity is just the usual matrix commutator.

²Savitri V. Iyer and S.G. Rajeev Mod.Phys.Lett.A17:1539-1550,2002[physics/0206083]





Then the regularized constants of motion are $Q_k = \frac{1}{N} \text{tr } \hat{\omega}^k$. The set of herimitean matrices with a fixed value of these constants has a finite volume, known from random matrix theory: $\prod_{k < l} (\lambda_k - \lambda_l)^2$, the λ_k being the eigenvalues. The entropy is thus $S = \frac{1}{N^2} \sum_{k \neq l} \log |\lambda_k - \lambda_l| = \mathcal{P} \int \log |\lambda - \lambda'| \rho(\lambda) \rho(\lambda')$ where $\rho(\lambda) d\lambda d\lambda' = \frac{1}{N} \sum_k \delta(\lambda - \lambda_k)$.

In the continuum limit this tends to a remarkably simple formula:

$$S = \mathcal{P} \int \log |\omega(x) - \omega(y)| d^2x d^2y \quad (18)$$

Even without a complete theory based on the Fokker-Plank equation (assuming that its continuum limit does exist) we can make some predictions about two dimensional turbulence.





The Shape of a Hurricane

What is the vorticity profile that maximizes entropy for fixed value of mean vorticity Q_1 and enstrophy³? This should be the most probable configuration for a vortex profile. Using a simple variational argument we get the answer in parametric form $\omega(r) = 2\sigma \sin \phi + \bar{Q}_1$, $r^2 = \frac{1}{2}[a_1^2 + a_2^2] \pm [a_2^2 - a_1^2] \frac{1}{\pi} [\phi + \frac{1}{2} \sin(2\phi)]$ in the region $a_1 \leq r \leq a_2$. We should expect this to be the vorticity distribution of the tornados and hurricanes.

³ $Q_2 = \sigma^2 + Q_1^2$



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Three Dimensional Turbulence

Main Question:

What is the 'quantum deformation' of the Lie algebra of volume preserving vector fields in three dimensions?



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Two Dimensional Nonlinear Models

A field $\phi : \Sigma \rightarrow M$ taking values in a Riemannian manifold (M, q) has action $S = \frac{1}{2} \int g^{\mu\nu} \sqrt{g} \partial_\mu \phi^a \partial_\nu \phi^b q_{ab} d^2x$ that is conformally invariant. Matrix regularization gives an interesting approach to these theories as well.

If we discretize space and keep time continuous, we will have a 'spin chain' whose configuration space is M^n . The hamiltonian can be chosen to be

$$H = \sum \Delta_i + \sum_i V(\phi_i, \phi'_{i+1}). \quad (19)$$



Here Δ is the Laplacian on (M, q) . The potential for nearest neighbors can be chosen to be just the square of the geodesic distance $V(\phi, \phi') = \lambda d(\phi, \phi')^2$. We should tune the coupling constant to a critical value, as $n \rightarrow \infty$ at which there is a second order phase transition to get the continuum limit.



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Short Range Interaction

Due to universality, there is a lot of freedom in the choice of this regularized hamiltonian; e.g., we could make other choices of $V(\phi, \phi')$ which reduce to the above for short distances. More importantly, we could allow interactions between sites that decay exponentially with distance on the chain:

$$H = \sum_i \Delta_i + \sum_{ij} N^{-|i-j|} V(\phi_i, \phi_j), \quad N > 1. \quad (20)$$





A Quantum Gas as a Regularization

If $N = 1$, this hamiltonian has a simple meaning: it is a many body system of n particles moving on the configuration space M subject to a pairwise potential $V(\phi, \phi')$. If we imagine now that each particle has a wavefunction valued in the space of $N \times N$ matrices and consider only states invariant under $U(N)$, we get exactly the above hamiltonian. We are *not* taking the limit as $N \rightarrow \infty$: all values of $N > 1$ should be in the same universality class.



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The interesting case is when M is of positive curvature and finite volume. Then as we let $n \rightarrow \infty$ we will approach a system of infinite density if we hold the volume fixed. There are logarithmic divergences and the metric needs to be renormalized (flattened out) to get a sensible limit. This gives an interesting new way of simulating a quantum field theory as well...

