Fluctuation and Dissipation in Fluids

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Fluid Mechanics

Fluids display many phenomena that remain puzzling to physicists. The motion of individual molecules in the fluid follows simple well-understood laws of classical mechanics. The collective motion of a large number of these molecules causes phenomena such as

1. **Turbulence.** Onset of turbulence is a kind of critical phenomenon—still poorly understood. Large scale disorder.

2. **Chaos.** Extreme sensitivity to initial conditions. “A butterfly flapping its wings in Australia can cause rain in New Jersey”

3. **Stable Vortices** How can hurricanes and the ‘Red Eye of Jupiter’—stable over long distances and times—arise out of chaos?
Macroscopic Variables

We need to give up on describing a fluid in terms of the position and momentum of each molecule, and instead pass to macroscopic variables such as pressure, density and velocity of fluid elements that contain averages over many molecules. This was done already in the eighteenth century by Leonhard Euler, “the Master of us all” mathematical physicists.

He applied this philosophy also to a rigid body, a collection of molecules that move together, keeping the distance between two of them fixed. There is a surprising mathematical similarity between these two systems, a connection that could not have escaped Euler. But modern geometric language allows us to express this ideas more clearly.
The Euler Equations

The Euler equations of an incompressible inviscid fluid are

\[
\frac{\partial}{\partial t} v + (v \cdot \nabla) v = -\nabla p, \quad \text{div } v = 0.
\]

We can eliminate pressure \( p \) by taking a curl. Defining the vorticity \( \omega = \text{curl } v \) and using \( (v \cdot \nabla) v = \omega \times v + \frac{1}{2} \nabla v^2 \), we get,

\[
\frac{\partial}{\partial t} \omega + \text{curl } [\omega \times v] = 0.
\]

We can regard \( \omega \) as the basic dynamic variable of the system, since \( v \) determined by it.

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\(^1\) We assume that the circulation \( \int_C v \) around con-contactible loops (if any) \( C \) are zero and that the boundary values of the velocity are given.
Groups of Transformations

The only ways to move a rigid body are to translate every particle in it by the same vector, or to rotate them all by the same angle round some axis. The composition of two such movements gives you another combination of a rotation and translation. Every such move has an inverse, bringing the body to its original configuration: the configuration space of the rigid body is a group, the *Euclidean group*. Infinitesimally, two vectors suffice to describe these motions: one for translation and another for rotation. Thus, the *configuration space of a rigid body is six dimensional*. 
The Diffeomorphism Group

The molecules of a fluid don’t have to be at a fixed distance from each other. Each point in the fluid \( x \) can be moved to another point \( \phi(x) \) as long as \( \phi \) is a differentiable function and also has a differentiable inverse. The result of composing two such transformations is another one, \( \phi_1(\phi_2(x)) \). Thus the configuration is also a group, the Diffeomorphism group of space in mathematical language: the time volution of the fluid is given by function \( \phi(x, t) \) that says where a fluid elements that was at \( x \) at time \( t = 0 \) will be found at later time \( t \).

This is an infinite dimensional space, since infinitesimally a transformation is described by a vector field \( v(x) \) that can be specified independently at each point.
Incompressible Flows

If the velocity of the fluid is small compared to the speed of sound in it, the density of the fluid cannot change. This imposes a constraint of incompressibility, weaker than that of a rigid body:

$$\det \frac{\partial \phi}{\partial x} = 1.$$ 

The velocity has to satisfy the infinitesimal version of this condition, \( \text{div} \ v = 0 \). There are still an infinite number of such vectors.

See V. I. Arnold and B. Khesin *Topological Methods in Hydrodynamics* for more on the geometrical picture of fluid flow.
Geodesic Motion

There is a natural way to measure the ‘size’ of an incompressible vector field:

$$||v||^2 = \int v^2(x)dx.$$  

It has a simple physical meaning as well: it is proportional to the total kinetic energy of the fluid. By breaking up any transformation as a composition of infinitesimal ones, this defines a notion of ‘distance’ between any configuration of the fluid and the original one (the identity element of the group). The distance between any two transformations $\phi_1, \phi_2$ can be defined as the distance of $\phi_1^{-1}\phi_2$ from the identity.

Euler’s equations simply state that the fluid always moves along the path of shortest distance in its configuration space!
Poisson Brackets

The Poisson bracket of two functions of vorticity is defined to be

\[ \{ F, G \} = \int \omega \cdot \left[ \text{curl} \frac{\delta F}{\delta \omega} \times \text{curl} \frac{\delta G}{\delta \omega} \right] d^3 x \]

This is the natural Poisson bracket on the dual of the Lie algebra of incompressible vector fields.

Using the identity \( \text{div} [a \times b] = b \cdot \text{curl} a - a \cdot \text{curl} b \) this may also be written as

\[ \{ F, G \} = \int \text{curl} \left[ \omega \times \text{curl} \frac{\delta F}{\delta \omega} \right] \cdot \frac{\delta G}{\delta \omega} d^3 x \]
The Hamiltonian Formalism

The hamiltonian is

\[ H = \frac{1}{2} \int v \cdot v \, d^3x. \]

Using the identity

\[ \text{curl} \, \frac{\delta F}{\delta \omega} = \frac{\delta F}{\delta v} \]

we get \( \{H, \omega\} = \text{curl} \, [\omega \times v] \) as needed to get the Euler equations.

The analogy of this to the equations of a rigid body

\[ \frac{dL}{dt} = L \times (I^{-1}L) \]

could not have escaped Euler. Vorticity is analogous to angular momentum; the Laplace operator is analogous to moment of inertia \( I \).
Sensitivity to Initial Conditions

In flat space, geodesics are straight lines: a small change in the initial point or direction only has a small change in the outcome. But if the space has negative curvature, geodesics that start out close depart from each other exponentially fast. (They focus towards each other for positive curvature, as in the case of great circles on a sphere).

The reason why hydrodynamics is so unstable is that the curvature of the metric of the configuration space is negative in almost all directions. This means that even small fluctuations in initial data cannot ever be ignored.
Entropy

We deal with such systems all the time in statistical mechanics. The basic idea is to look for ‘macroscopic’ variables (density, pressure etcd.) that are averages over large numbers of particles so that fluctuations average out. But then we have to live with partial information: with entropy. The most likely state of the system will be one that maximizes this entropy for a given value of the conserved quantities.

Is there a similar notion of entropy for turbulent fluctuations?
Entropy for Two Dimensional Flow


We found a remarkably simple formula that measures the turbulent entropy for two dimensional incompressible flow:

$$\int \log |\omega(x) - \omega(y)| dxdy.$$ 

We don’t have one yet for three dimensional flow yet.
The Profile of a Hurricane

What is the vorticity profile that maximizes entropy for fixed value of mean vorticity $Q_1$ and enstrophy $^2Q_2$? This should be the most probable configuration for a vortex profile. Using a simple variational argument we get the answer in parametric form

$$\omega(r) = 2\sigma \sin \phi + \bar{Q}_1, \quad r^2 = \frac{1}{2}[a_1^2 + a_2^2] \pm [a_2^2 - a_1^2] \frac{1}{\pi} \left[ \phi + \frac{1}{2} \sin (2\phi) \right]$$

in the region $a_1 \leq r \leq a_2$. We should expect this to be the vorticity distribution of the tornados and hurricanes.

\[^2Q_2 = \sigma^2 + Q_1^2\]
Notice the ‘eye’ of the hurricane: the flow is fastest right inside the eyewall and then decreases gradually to the outer wall.
Reverse Cascade

In three dimensions, vortices tend to break up into smaller scales, leading to very complicated behavior (the ‘cascade’ effect). In two dimensional flow, the opposite can be observed: vortices tend to combine into a few big ones. We have a simple explanation for this phenomenon: The entropy of vortices increases as they combine!

The hurricane is stable in a chaotic environment because it maximises the entropy.
Two-Dimensional Euler Equations

It is useful to look first at the much simpler example of two-dimensional incompressible flow.

Area preserving transformations are the same as canonical transformations. The corresponding Lie algebra is the algebra of symplectic transformations on the plane:

$$[f_1, f_2] = \epsilon^{ab} \partial_a f_1 \partial_b f_2.$$ 

The above Lie bracket of functions gives the Poisson bracket for vorticity:

$$\{\omega(x), \omega(y)\} = \epsilon^{ab} \partial_b \omega(x) \partial_a \delta(x - y).$$
Hamiltonian Formulation

Euler equations follow from these if we postulate the hamiltonian to be the total energy of the fluid,

$$H = \frac{1}{2} \int u^2 d^2 x = \frac{1}{2} \int G(x, y)\omega(x)\omega(y)d^2 x d^2 y.$$

The quantities $Q_k = \int \omega^k(x)d^2 x$ are conserved for any $k = 1, 2 \cdots$: these are the Casimir invariants. In spite of the infinite number of conservation laws, two dimensional fluid flow is chaotic.
Lie Algebra of Canonical Transformations

The Lie algebra of symplectic transformations can be thought of as the limit of the unitary Lie algebra as the rank goes to infinity. To see this, impose periodic boundary conditions and write in terms of Fourier coefficients as

\[ H = (L_1 L_2)^2 \sum_{m \neq (0,0)} \frac{1}{m^2} |\omega_m|^2, \quad \{\omega_m, \omega_n\} = -\frac{2\pi}{L_1 L_2} \epsilon_{ab} m_a n_b \omega_{m+n}. \]
Non-Commutative Regularization

Using an idea of Fairlie and Zachos, we now truncate this system by imposing a discrete periodicity mod $N$ in the Fourier index $m$; the structure constants must be modified to preserve periodicity and the Jacobi identity:

$$\{\omega_m, \omega_n\} = \frac{1}{\theta} \sin[\theta(m_1 n_2 - m_2 n_1)]\omega_{m+n} \mod N, \quad \theta = \frac{2\pi}{N}.$$ 

This can be thought of as a 'quantum deformation' of the Lie algebra of symplectic transformations. This is the Lie algebra of $U(N)$. 
Regularized hamiltonian

The hamiltonian also has a periodic truncation

\[ H = \frac{1}{2} \sum_{\lambda(m) \neq 0} \frac{1}{\lambda(m)} |\omega_m|^2, \]

\[ \lambda(m) = \left\{ \frac{N}{2\pi} \sin \left[ \frac{2\pi}{N} m_1 \right] \right\}^2 + \left\{ \frac{N}{2\pi} \sin \left[ \frac{2\pi}{N} m_2 \right] \right\}^2. \]

This hamiltonian with the above Poisson brackets describe the geodesics on \( U(N) \) with respect to an anisotropic metric. We are writing it in a basis in which the metric is diagonal.
Entropy of Matrix Models

The constants $Q_m = \int \omega^m(x) d^2x$ define some infinite dimensional surface in the space of all functions on the plane. The micro-canonical entropy (a la Boltzmann) of this system will be the log of the volume of this surface. How to define this volume of an infinite dimensional manifold? We can compute it in the regularization and take the limit. It is convenient to regard $\omega$ as a matrix by defining $\hat{\omega} = \sum_m \omega_m U(m)$ where $U(m) = U_1^{m_1} U_2^{m_2}$ with the defining relations $U_1 U_2 = e^{\frac{2\pi i}{N}} U_2 U_1$. In this basis, the Lie bracket of vorticity is just the usual matrix commutator.

Then the regularized constants of motion are $Q_k = \frac{1}{N} \text{tr} \hat{\omega}^k$. The set of hermitian matrices with a fixed value of these constants has a finite volume, known from random matrix theory: $\prod_{k<l} (\lambda_k - \lambda_l)^2$, the $\lambda_k$ being the eigenvalues.
Entropy of Turbulent Flow

The entropy is thus

\[ S = \frac{1}{N^2} \sum_{k \neq l} \log |\lambda_k - \lambda_l| = \mathcal{P} \int \log |\lambda - \lambda'| \rho(\lambda) \rho(\lambda') \]

where \( \rho(\lambda) d\lambda d\lambda' = \frac{1}{N} \sum_k \delta(\lambda - \lambda_k) \).

In the continuum limit this tends to a remarkably simple formula:

\[ S = \mathcal{P} \int \log |\omega(x) - \omega(y)| d^2x d^2y \]

Even without a complete theory based on the Fokker-Plank equation (assuming that its continuum limit does exist) we can make some predictions about two dimensional turbulence.
Back to 3D: Clebsch Variables

The Poisson brackets are rather complicated in terms of vorticity. It was noticed by Clebsch that any solution to \( \text{div } \omega = 0 \) can be written as

\[
\omega = \nabla \lambda \times \nabla \mu
\]

for a pair of functions \( \lambda, \mu : M \rightarrow R \). The advantage of this parametrization is that the Poisson brackets of vorticity follow from canonical commutation relations of these functions:

\[
\{ \lambda(x), \mu(y) \} = \delta(x, y), \quad \{ \lambda(x), \lambda(y) \} = 0 = \{ \mu(x), \mu(y) \}.
\]
Statistical Field Theory of Turbulence

The Euler equations describe the geodesics in the group of volume preserving diffeomorphisms. Arnold showed that the sectional curvature of this metric is negative in all but a finite number of directions. This means that the solutions are very unstable: small errors in the initial conditions will grow exponentially: explaining the unpredictability of fluid flow. In other words, we cannot ignore fluctuations in the external forces acting on the fluid, as their effect will grow exponentially with time. This could well be the cause of turbulence.

Such systems should not really be thought of as deterministic. We are led seek a theory analogous to statistical mechanics of gases, in which the velocity is a random field whose probability distribution is given by an analogue of the Fokker-Plank equation.
Fluctuations

We can model these turbulent fluctuations by adding a Gaussian random force field to the r.h.s.

\[
\frac{\partial}{\partial t} v + (v \cdot \nabla)v = -\nabla p + f, \quad \Rightarrow \quad \frac{\partial}{\partial t} \omega + \text{curl} [\omega \times v] = \text{curl} f
\]

What would be a good choice of covariance for this Gaussian? \(< f^i(x, t) >= 0, < f^i(x, s) f^j(y, t) > = \delta(s-t) G_{ij}(x, y) >\)

We recall that a system with fluctuations must always have dissipation: otherwise the energy pumped into the system through fluctuations will grow without bound and we will get infinite temperature: temperature is the ratio of fluctuation to dissipation. Although there isn’t as yet a universally accepted model for statistical fluctuations in turbulence, there is one for dissipation. We will use it along with a principle of detailed balance to get a model for fluctuations.
Viscosity

The Navier-Stokes equations of hydrodynamics allow for dissipation of energy through internal friction (viscosity):

\[
\frac{\partial}{\partial t} v + (v \cdot \nabla) v = \nu \nabla^2 v - \nabla p \Rightarrow \frac{\partial}{\partial t} \omega + \text{curl} [\omega \times v] = \nu \nabla^2 \omega.
\]

What should be the covariance of fluctuation and dissipation in order that the average energy pumped in fluctuation is dissipated away?
A Geometric Model of Dissipation

Let $\Omega^{ij}$ be the Poisson tensor (inverse of a symplectic form $\Omega_{ij}$) and $H$ the Hamiltonian of some system with a finite number of degrees of freedom, for simplicity. Then the Hamilton’s equations are

$$\frac{d\xi^i}{dt} = \Omega^{ij} \partial_j H.$$  

A nice model for dissipation is to add the negative gradient of the Hamiltonian with respect to some metric to the r.h.s.:

$$\frac{d\xi^i}{dt} = \Omega^{ij} \partial_j H - g^{ij} \partial_j H \Rightarrow \frac{dH}{dt} = -g^{ij} \partial_i H \partial_j H \leq 0.$$  

The Navier-Stokes equations fit this mold except that it is infinite dimensional; the role of the metric $g^{ij}$ is played by the square of the Laplace operator.
A Geometric Model of Fluctuations

Suppose we add a random force to the r.h.s.

\[
\frac{d\xi^i}{dt} = \Omega^{ij} \partial_j H - g^{ij} \partial_j H + \eta^i
\]

with zero mean and Gaussian covariance

\[
< \eta^i(\xi, t) \eta^j(\xi', t') > = \delta(t - t') \delta(\xi, \xi') G^{ij}(\xi).
\]

This stochastic equation (Langevin equation) is equivalent to the Fokker-Plank equation for the probability density \(W\):

\[
\frac{\partial W}{\partial t} + \partial_i \left( \left[ \Omega^{ij} \partial_j H - g^{ij} \partial_j H \right] W \right) = \partial_i \left[ \sqrt{G} \ G^{ij} \partial_j \left( \frac{W}{\sqrt{G}} \right) \right]
\]

Note that \(\int W d\xi\) is conserved.
The Fluctuation-Dissipation Theorem

See S. Chandrashekhar Rev. Mod. Phys. 1941 for the basic idea.

Let us seek a static solution of this equation. The fluctuation and dissipation terms will balance each other out if

$$G^{ij} \sqrt{G} \frac{\partial_i \left( \frac{W}{\sqrt{G}} \right)}{W} = -g^{ij} \partial_j H;$$

that is, i.e.,

$$\partial_i \log \left[ \frac{W}{\sqrt{G}} \right] = -G_{ij} g^{jk} \partial_k H. \text{ The Boltzmann distribution } W = e^{-\beta H} \sqrt{\det G} \text{ will satisfy this condition if }$$

$$\beta G^{ij} = g^{ij}$$

The remaining term in the Fokker-Plank equation will also vanish if $\sqrt{G} = \gamma \text{Pf}\Omega$ so that the volume defined by the metrics agree with that defined by the symplectic form: $\partial_i \left[ \Omega^{ij} \partial_j e^{-\beta H} \text{Pf}\Omega \right] = 0$ is obvious in the Darboux co-ordinates.
The Covariance Metric for Turbulence

We saw earlier that the phase space of an incompressible fluid is the space of vorticities; i.e., vector fields satisfying \( \text{div} \, \omega = 0 \). The dissipation term is

\[
\nu \nabla^2 \omega = \nu \nabla^2 \text{curl} \, v = \nu \nabla^2 \text{curl} \, \text{curl} \, \frac{\delta H}{\delta \omega} = -\nu \nabla^4 \frac{\delta H}{\delta \omega}.
\]

Thus the tensor \( g^{ij} \) above corresponds to the square of the Laplace operator: \( \nabla^4 \). In order the energy pumped into the turbulent flow by fluctuations be dissipated by viscous damping, we will need the covariance of the fluctuations must be:

\[
< \eta^i(x) \eta^j(y) > = \beta \nu \delta^{ij} \nabla^4 \delta(x, y).
\]
Regularization

To make sense of such singular stochastic equations we will need a regularization, replacing the system with an infinite number of degrees of freedom by a sequence approximations by finite dimensional systems. Such discretizations are necessary anyway in the numerical solutions of hydrodynamics equations. The main mathematical difficulty is in maintaining the structure of the phase space as a Lie algebra. In the case of two dimensional incompressible flow we found a regularization by a matrix model.

We will need both a cut-off in real space (so that the system has finite volume) as well as a cut-off in momentum space (so that the kinetic energies of the advected particles is finite) in order to get a system with a finite number of degrees of freedom.
Finite Distance Cut-off

The first cut-off to finite volume is easy to do within conventional differential geometry: we can modify the metric on $R^3$ so that any two points are at a finite distance from each other. For example, if $ds^2 = h_{ij} dx^i dx^j = \frac{dx^2}{\left[1+\frac{x^2}{R^2}\right]^2}$ we get a sphere $S^3$. The canonical commutation pairing $<\lambda, \mu> = \int \lambda(x)\mu(y) \sqrt{h}d^3x$ continues to make sense and so does the hamiltonian

$$H = \frac{1}{2} \int \omega^i(x)G(x-y)_{ij}\omega^j(y)dxdy$$

where $\omega^i = \epsilon^{ijk}\partial_j\lambda\partial_j\mu$ and $G^{ij}(x,y)$ is the Greens function of the Laplace operator on vector densities. The equations of motion for $\lambda, \mu$ (or even for $\omega$) follow from this hamiltonian and the above canonical pairing.
The Quantum Three-Sphere

To reduce the number of degrees of freedom further and get a system with a finite number of degrees of freedom, we need to deform this further. There is a well-known finite dimensional deformation of space of functions on the three-sphere, the algebra $\text{Fun}_q(S^3)$ of functions on the ‘quantum group’ $SU(2)_q$ for $q = e^{i\pi/N}$. The word ‘quantum’ means here simply that this deformation is non-commutative: it does not mean that we are studying any quantum mechanical effects in the physical sense. The price we pay for having a cut-off in both real space and momentum space is this non-commutativity. See for example A. Jevicki, M. Mikailov and S. Ramgoolam hep-th/0008186; A Guide to Quantum Groups by V. Chari and A. Pressley.
Generators and Relations

For integer $N$ and $q = e^{i\pi N}$, $Fun_q(S^3)$ is the associative algebra generated by $\alpha, \beta, \alpha^*, \beta^*$ with relations

\[
\begin{align*}
\alpha\beta &= q\beta\alpha, & \alpha\beta^* &= q\beta^*\alpha, \\
[\alpha, \alpha^*] &= (q^{-1} - q)\beta\beta^*, & \beta^* &= \beta^*\beta, & \beta\alpha^* &= q\alpha^*\beta, \\
\alpha^{2N} &= \beta^{2N}, & \beta^{2N} &= \beta^{2N}, & \alpha^{2N} &= \beta^{2N}, & \alpha^{2N} &= \beta^{2N} = 1.
\end{align*}
\]

This is the deformation of the algebra of functions on $SU(2) = S^3$. In the book by Chari and Pressley (section 7.4C) there is given a quantum differential calculus on this space.
Non-Commutative Regularization of Hydrodynamics

Together this allows us to make sense of a generalization of the Clebsch bracket $d\lambda \wedge d\mu$. We can expand these differential forms in terms of representations of the quantum group; only representations of ‘spin’ $< N - 1$ will appear. The Laplacian has the irreducible representations as eigenspaces and eigenvalues that are given by a simple deformation of the usual formula.

The integral of a function (quantum analogue of the Haar measure) is simply given by the projection to the trivial representation. Thus we have all the ingredients needed to make sense of the Hamiltonian and the Poisson bracket: $\lambda$ and $\mu$ are finite dimensional hermitean matrices representing the above algebra.