PHY 254 Nuclear and Particle Physics, U. of Rochester

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Chapter 1

Rutherford Scattering

1.1 We will study the problem of a stream of alpha-particles scattered by a point-like nucleus. The main aim is to calculate the number of particles scattered into a given angle.

1.1.1 Since the source of particles is far away we can assume they are moving in the same direction. We can chose the co-ordinate system so that the scatterer (the nucleus) is at the center and the source is at infinity along the negative x-axis. There is a flux of N particles per unit time, per unit area normal to the x-axis. We want to know the number dn of particles scattered per unit time to a solid angle $d\Omega$. Clearly this will be proportional to N, so it is the ratio $d\sigma = \frac{dn}{N}$ that matters. This quantity has the dimension of area (because N is the number *per unit area*) and is called the differential scattering cross section.

1.1.2 Imagine an annular region circle of radius between b and $b+db$ normal to the x-axis and far away from the scatterer. There will be $N 2\pi b db$ particles coming in through this region. They will follow some trajectory and end up at infinity between angles θ and $\theta + d\theta$. The solid angle where they will appear is $d\Omega = 2\pi \sin\theta d\theta$. Thus the differential scattering cross section

$$\frac{d\sigma}{d\Omega} = \frac{1}{\sin\theta} b \left|\frac{db}{d\theta}\right|.$$

We take the absolute value because the derivative is actually negative: farther particles are scattered through smaller angles.

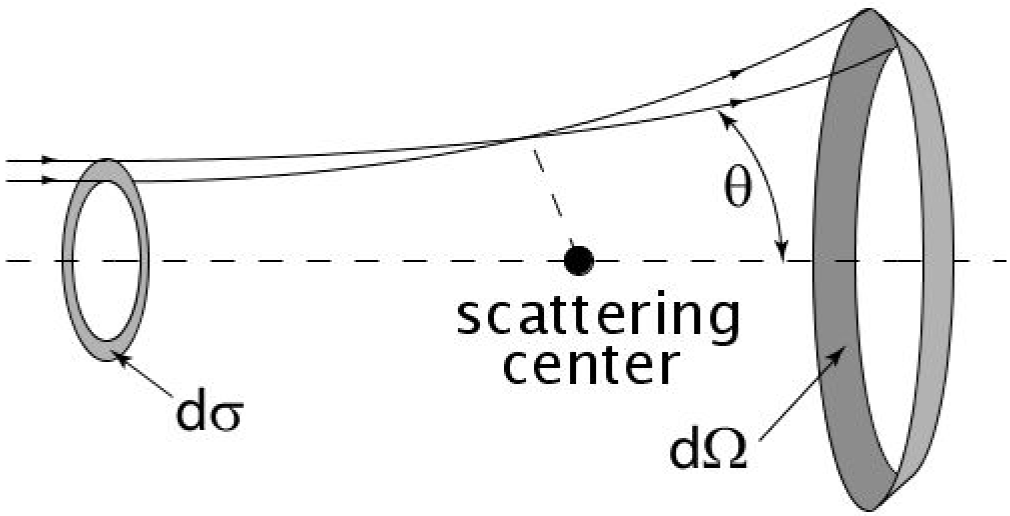

Figure 1.1: The Scattering Angle

1.1.3 This is a very important idea in all of nuclear and particle physics, so it is good to understand it for a simpler case than the Coulomb potential.

1.2 A hard sphere of radius a is a potential that is zero for $r > a$ and infinite for $r < a$.

1.2.1 Thus particles that are outside a distance a are not affected at all while particles that get to a distance a are reflected back with the same kinetic energy. The component of momentum parallel to the reflecting surface is not changed: there is no force in that direction. Conservation of energy then implies that the component normal to the surface simply changes sign. This means that the angle of incidence i (the angle that the incoming particle makes with the normal) is equal to the angle of reflection r. Much like in geometrical optics.

1.2.2 A little geometry will show that $i + r + \theta = \pi$ and that $b = a \sin i$. Since $i = r$, we get $b = a \cos \frac{\theta}{2}$.

1.2.3 Thus for the hard sphere

$$\frac{d\sigma}{d\Omega} = b\frac{db}{\sin\theta d\theta} = \frac{1}{4}a^2.$$

It is a special property of the hard sphere that the number of particles scattered into a solid angle is independent of θ.

1.2.4 Note that the total scattering cross-section

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \pi a^2$$

is simply that the cross-sectional area of the sphere. This is why the term 'scattering cross-section' is appropriate for the quantity that measures the scattering probability.

1.3 How much can we generalize the above discussion?

1.3.1 As long as particles are being reflected by some surface of revolution, the geometry we described above is still valid. It is still true that the differential cross-section is

$$d\sigma = 2\pi b db = 2\pi b \left| \frac{db}{d\theta} \right| d\theta$$

For a convex curve (i.e.,, a straight line cuts it at no more than two points) b is a monotonically decreasing function. The total cross section will be

$$\sigma = \int_0^\pi \frac{d\sigma}{d\theta} d\theta = \pi \left[b^2 \right]_\pi^0 = \pi a^2$$

where a is the largest value of b, at which $\theta = 0$. This is the point on the surface at which the distance measured normal to the beam axis is greatest. Thus the total scattering cross section is still the cross sectional area of the reflecting surface.

1.3.2 What is the total scattering cross section for a surface of revolution obtained by rotating a curve that is not convex? e.g., shaped like an apple.

1.3.3 Aside: The same analysis applies to the reflection of radio waves by an aircraft. Stealth technology is about figuring out shapes which have almost zero scattering cross-section in directions where the detectors are likely to be located; e.g., in standard radar the source and detector are at the same location. Diffraction effects and absorption can also be used to reduce the radar cross-section.

1.3.4 Quantum mechanics predicts corrections to all this. More on that later.

1.4 Let us now turn away from the practice problem to the real problem of Rutherford scattering.

1.4.1 Given the energy E and the angular momentum L the path of the particle in a central potential $V(r)$ is given by

$$\frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + V(r) = E, \quad mr^2\dot{\phi} = L.$$

At infinity, the velocity v is related to E and L by

$$\frac{1}{2}mv^2 = E, \quad mvb = L$$

where b is the impact parameter, or the distance that the particle is away from the beam axis. Thus

$$\frac{L^2}{2m} = Eb^2$$

A little algebra will give

$$d\phi = \frac{bdu}{\left[1 - \frac{V(r)}{E} - b^2u^2\right]^{\frac{1}{2}}}, \quad r = \frac{1}{u}.$$

Integrating this gives the polar angle as a function of the inverse radius; i.e., the trajectory of the particle.

1.4.2 In the beginning r is large and $\phi = 0$; in the midpoint r takes its least value and eventually r grows to infinity again and ϕ tends to some value $\pi - \theta$ where θ is the scattering angle.

1.4.3 It is not really necessary to know the whole trajectory, only the change in direction of the particle over the whole path. This is twice the change in direction from the point of closest approach, where u takes its maximum value u_1. That is, $\theta = \pi - 2\phi_0$ where

$$\phi_0 = \int_0^{u_1} \frac{bdu}{\left[1 - \frac{V(r(u))}{E} - b^2u^2\right]^{\frac{1}{2}}}$$

1.4.4 The maximum u_1 is the positive root of the polynomial under the square root sign. For, that is where the quantity $\frac{du}{d\phi}$ would vanish, which is the denominator, would vanish.

1.4.5 For a repulsive Coulomb potential

$$V(r) = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0}\frac{1}{r}.$$

Define a constant with the dimension of length

$$a = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 E}.$$

This is the distance of closest approach by any incoming particle.

1.4.6 Then

$$\phi_0 = \int_0^{u_1} \frac{b du}{[1 - au - b^2u^2]^{\frac{1}{2}}}.$$

1.4.7 The integral is elementary and can be done by a trigonometric substitution. (Or you can look it up in a table of integrals.) The answer is

$$\phi_0 = \frac{\pi}{2} - \arcsin\frac{1}{\sqrt{\left[1 + \frac{4b^2}{a^2}\right]}}$$

so that

$$1 + \frac{4b^2}{a^2} = \frac{1}{\sin^2\frac{\theta}{2}}.$$

1.4.7.1 We digress to explain the evaluation of the integral. Any quadratic function $Au^2 + Bu + C$ can be brought to the form

$$Au^2 + Bu + C = A(u - u_1)(u - u_2)$$

where $u_{1,2} = \frac{-B \pm \sqrt{(B^2 - 4AC)}}{2A}$ are its roots. If the discriminant $\Delta = B^2 - 4AC > 0$ (which is our case) both roots are real. It is convenient first to introduce the variable $v = u + \frac{B}{2A}$ so that

$$Au^2 + Bu + C = (-A)(\frac{\Delta}{4A^2} - v^2).$$

We pull out $-A$ rather than A because A is negative in our case. The integral we want is of the form

$$\int_{u_0}^{u_1} \frac{du}{\sqrt{[Au^2+Bu+C]}}$$

In terms of v , it becomes

$$\frac{1}{\sqrt{(-A)}}\int_{u_0+\frac{B}{2A}}^{\sqrt{\Delta}} \frac{dv}{\sqrt{(\frac{\Delta}{4A^2}-v^2)}}.$$

1.4.7.2 Now put $v = \frac{\sqrt{\Delta}}{2|A|}\sin\phi$ so that $dv = \frac{\sqrt{\Delta}}{2|A|}\cos\phi d\phi$ and $\sqrt{(\frac{\Delta}{4A^2}-v^2)]} = \sqrt{\frac{\Delta}{2|A|}}\cos\phi$. The limits of integrals become $\frac{\pi}{2}$ and $\arcsin\frac{2|A|(u_0+\frac{B}{2A})}{\sqrt{\Delta}} = \arcsin\frac{2|A|u_0-B}{\sqrt{\Delta}}$ respectively. Thus the integral becomes

$$\int_{u_0}^{u_1} \frac{du}{\sqrt{[Au^2+Bu+C]}} = \frac{1}{\sqrt{|A|}}\left[\frac{\pi}{2}-\arcsin\frac{[2|A|u_0-B]}{\sqrt{\Delta}}\right]$$

Now we just put in $u_0 = 0, A = -b^2, B = -a, \Delta = a^2+4b^2$ to get the answer.

1.4.8 The differential cross-section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{2\sin\theta}\left|\frac{db^2}{d\theta}\right| = \frac{a^2}{16}\frac{1}{\sin^4\frac{\theta}{2}}.$$

which is Rutherford's famous result.

1.4.9 The total cross section, integrated over all angles is infinite. Unlike the hard sphere, the Coulomb force can in principle affect a particle that is far away. In practice, the scattering cannot be observed for small angles: the intensity of the incoming beam would overwhelm any scattering signal.

1.4.10 The target has many electrons as well. Why is it correct to ignore their effect on the alpha particles? How would you take into account of the recoil of the nucleus?

1.4.11 How would this change when relativistic effects are added? How would you calculate this in the quantum theory?

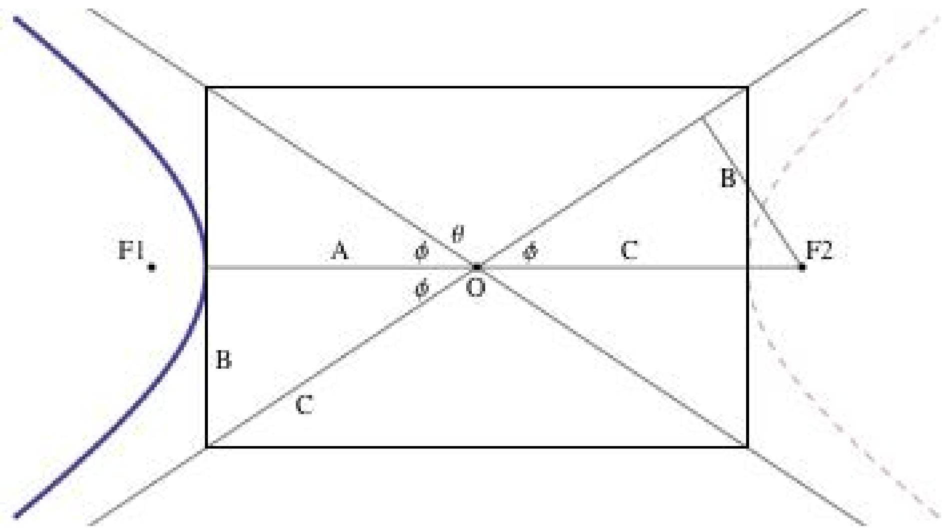

Figure 1.2: Geometry of Rutherford Scattering

1.1 A Geometric Derivation of Rutherford's Formula

One you know that the trajectory of a particle in a Coulomb potential is a hyperbola, we can derive the Rutherford formula using geometry. The equation of a hyperbola with semi-major axis A and semi-minor axis B is

$$\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1.$$

This has two connected components, one of which is the orbit. The nucleus is located at one the focii, say F2 in the above diagram. The distance of F2 from the origin is $C = \sqrt{A^2 + B^2}$.

For a repulsive potential the trajectory will be the blue curve and for an attractive potential it will be the dashed curve. In the case of alpha particles, the potential is repulsive.

The asymptotes are the lines passing through the origin with slopes $\pm\frac{B}{A}$. The scattering angle is the angle between the incoming and outgoing directions. It is just the angle between asymptotes, which we denote by θ in the diagram.

The impact parameter is the distance from to the initial asymptote. From the two similar right angled triangles in the diagram, we can see that the impact parameter is the same as the semi-minor axis B .

$$b = B.$$

Also from the diagram we see that

$$\theta + 2\phi = \pi,$$

$$\tan\phi = \frac{B}{A}$$

so that

$$B = A\cot\frac{\theta}{2}.$$

It remains to determine the semi-major axis A in terms of the physical parameters. The distance of closest approach of this trajectory to the nucleus is $A + C = D$ (say). At this point the radial velocity (with respect to the nucleus) is zero. Thus the kinetic energy is all from angular momentum. Adding it to the potential energy we get the conserved total energy

$$E = \frac{L^2}{2mD^2} + \frac{K}{D}$$

where $K = \frac{Z_1 Z_2}{4\pi\epsilon_0}$ is the constant that appears in the potential energy.

At infinity the potential energy vanishes; the angular momentum and energy are (as in the text)

$$E = \frac{1}{2}mv^2, \quad L = mvb, \Rightarrow \frac{L^2}{2m} = Eb^2.$$

Thus

$$E\frac{B^2}{D^2} + \frac{K}{D} = E$$

or

$$D^2 - \frac{K}{E}D - B^2 = 0$$

But remember that

$$D = A + \sqrt{A^2 + B^2}$$

Putting this into the quadratic equation for A ,and after a little algebra

$$A = \frac{K}{2E}.$$

Thus,

$$b = \frac{K}{2E} \cot \frac{\theta}{2}$$

This is as far geometry will take us.

Now,we just have to put this in and differentiate:

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| \\
&= \frac{1}{2\sin\theta}\left|\frac{db^2}{d\theta}\right| \\
&= \frac{K^2}{4E^2}\,\frac{1}{2\sin\theta}\,\left|\frac{d}{d\theta}\left[\cot^2\frac{\theta}{2}\right]\right| \\
&= \frac{K^2}{4E^2}\,\frac{1}{2\sin\theta}\,\left|\frac{d}{d\theta}\left[\sin^{-2}\frac{\theta}{2}\right]\right| \\
&= \frac{K^2}{4E^2}\,\frac{1}{2\sin\theta}\,\frac{\cos\frac{\theta}{2}}{\sin^3\frac{\theta}{2}}
\end{aligned}$$

and finally

$$\frac{d\sigma}{d\Omega} = \frac{K^2}{16E^2}\,\frac{1}{\sin^4\frac{\theta}{2}}$$

1.2 Appendix:Finite Nuclear Size

1.2.1 The potential

We showed that for a monotonic central potential $V(r)$ the scattering angle $\theta = \pi - 2\phi_0$ is given by

$$\phi = \int_0^{u_1} \frac{b\,du}{\sqrt{1 - \frac{V(r(u))}{E} - b^2u^2}}.$$

Here u_1 is the inverse of the distance of closest approach to the center at which

$$1 - \frac{V(r(u))}{E} - b^2u^2 = 0.$$

The potential we are interested in now is

$$V(r) = \begin{cases} \frac{Z_1Z_2e^2}{4\pi\epsilon_0 r} \text{ for } r > r_0 \\ 0 \text{ for } r < r_0 \end{cases}$$

Put

$$\frac{Z_1Z_2e^2}{4\pi\epsilon_0 a} = E.$$

so that

$$V(r) = \begin{cases} \frac{a}{r} \text{ for } r > r_0 \\ 0 \text{ for } r < r_0 \end{cases}$$

1.2.2 Large Impact Parameter

Thus as long as distance of closest approach is greater than r_0 the orbits will be hyperbolic. If $a > r_0$ this is true for all scattering angles. Even if $a < r_0$, the orbits remains hyperbolic for those angles and impact parameters for which the positive root u_1 of

$$1 - au - bu^2 = 0$$

is smaller than $u_0 = \frac{1}{r_0}$. Now,

$$u_1 = \frac{a - \sqrt{a^2 + 4b^2}}{-2b^2} = \frac{\sqrt{a^2 + 4b^2} - a}{2b^2}$$

Thus $u_1 < u_0$ becomes

$$\sqrt{a^2 + 4b^2} < a + 2b^2u_0$$

Or, after some algebra

$$b > b_0 \equiv \sqrt{r_0(r_0 - a)}.$$

Using the relation

$$1+\frac{4b^2}{a^2}=\frac{1}{\sin^2\frac{\theta}{2}},\quad \frac{2b}{a}=\cot\frac{\theta}{2}$$

we see that this critical impact parameter b_0 corresponds to an angle $\theta > \theta_0$ given by

$$\cot\frac{\theta_0}{2}=2\sqrt{\frac{r_0}{a}\left(\frac{r_0}{a}-1\right)}$$

1.2.3 Small Impact Parameter

Ir remains to understand the orbit when $b < b_0$. Clearly the particle will move along a hyperbola as long as $r > r_0$. Inside the nucleus, $r < r_0$ it will move along a straight-line, since the potential is zero. This means that the scattering angle of trajectories with $b < b_0$ is *smaller* than those with $b > b_0$. *The scattering angle cannot be any larger than* θ_0 *.*

To understand this think of the trajectory with $b = 0$. This head-on collision (with large enough energy that $a < r_0$) will get the particle inside the nucleus; it will move on a straightline to the opposite side and emerge in the forward direction. Hence, for $b = 0$, $\theta = 0$ instead of $\theta = \pi$ as with a point-like target!

Thus both the regions $b > b_0$ and $b < b_0$ get mapped to the range of scattering angles $\theta_0 > \theta > 0$. The cross-section in angles greater than θ_0 vanishes.

$$\frac{d\sigma}{d\Omega}=0,\qquad \text{for } \theta > \theta_0.$$

For the same $\theta < \theta_0$, there will be two different impact parameters, one with $b(\theta) < b_0$ and the other greater than $b_2(\theta) > b_0$. (b is a double valued function of θ).The differential cross-section will be the sum of the two contributions

$$\frac{d\sigma}{d\Omega}=\frac{1}{\sin\theta}\left[b\left|\frac{db}{d\theta}\right|+b_2\left|\frac{db_2}{d\theta}\right|\right]$$

We have,

$$\phi_0 = \int_0^{u_0} \frac{bdu}{\sqrt{1 - au - b^2u^2}} + \int_{u_0}^{b^{-1}} \frac{bdu}{\sqrt{1 - b^2u^2}}$$

Since $\frac{\theta}{2} = \frac{\pi}{2} - \phi_0$ and evaluating the second integral

$$\frac{\theta}{2} = \arcsin bu_0 - \int_0^{u_0} \frac{bdu}{\sqrt{1 - au - b^2u^2}}$$

We need this for small b .

$$\frac{\theta}{2} \approx bu_0 + \frac{2b}{a}[\sqrt{1 - au_0} - 1] + \mathrm{O}(b^2)$$

Then

$$\frac{db}{d\theta} \approx \frac{a}{2}[au_0 + 2\left(\sqrt{1 - au_0} - -1\right)]^{-1} + \mathrm{O}(\theta)$$

Thus

$$\left[\frac{d\sigma}{d\Omega}\right]_{\text{inside}} = \frac{1}{\sin\theta} b(\theta) \left|\frac{db}{d\theta}\right| \approx \frac{a^2}{4}[au_0 + 2\left(\sqrt{1 - au_0} - 1\right)]^{-2} + \mathrm{O}(\theta)$$

This is a small (i.e., finite) correction to the forward scattering due to hyp;erbolic trajectories,

$$\left[\frac{d\sigma}{d\Omega}\right]_{\text{ouside}} \approx \frac{a^2}{\theta^4} + \cdots$$

which is divergent. Thus the particles that get inside the nucleus will be lost into outgoing the beam and will be hard to see: the correction is down by four powers of θ .

1.2.4 Conclusion

The main effect of a finite size is that there is a maximum scattering angle above which the differential cross section is just zero;i.e., no particles will be found with $\theta > \theta_0$. From this maximum angle we can determine the nuclear radius r_0 as a multiple of the parameter a which is related to energy:

$$\cot \frac{\theta_0}{2} = 2\sqrt{\frac{r_0}{a}\left(\frac{r_0}{a} - 1\right)}$$

and

$$a = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 E}$$

Chapter 2

Why We Need Quantum Mechanics

2.1 Quantum effects cannot be ignored in most of atomic and nuclear physics. But relativistic effects are small.

2.2 Both relativity and quantum mechanics come into play in particle physics.

2.2.1 Quantum effects become important if the de Broglie wavelength $\lambda = \frac{h}{p}$ (where p is the momentum)is of the order of the size of the system.

2.2.2 Quantum effects kick in when we reach the size of an atom. The energy E of a valence electron in an atom is of the order an electron volt; the mass of the electron is about $\frac{10^6 \text{eV}}{c^2}$. Thus its momentum $p = \sqrt{[2Em]} \sim 10^3 \frac{\text{eV}}{c}$ .Plank's constant times the velocity of light is $hc \sim 10^{-6}$ eVm. Thus the de Broglie wavelength $\lambda = \frac{hc}{pc}$ is about a nanometer. Thus any smaller than a nanometer will involve quantum effects. An atom is about a tenth to about a half of a nanometer in size.

2.2.3 The size of a nucleus is even smaller, about one Fermi $\sim 10^{-15}$ m. This corresponds to a momentum of about $10 \ \frac{\text{MeV}}{\text{c}}$. An electron with this momentum would be highly relativistic, so even outside the scope of non-relativistic quantum mechanics. But protons and neutrons with this much momentum can be treated within non-relativistic quantum mechanics. That is because their mass is of the order of $10^3 \ \frac{MeV}{c^2}$.

2.2.4 The binding energy of a proton in the simplest nucleus (deuterium) is about 2 MeV. Thus we can hope to understand nuclear physics in terms of non-relativistic quantum mechanics.

2.2.5 Even though an alpha particle is a bound state of two neutrons and two protons, it is so tightly bound that in some heavier nuclei it can be thought of as an 'elementary' constituent of the nucleus.

Chapter 3

The Axioms of Quantum Mechanics

3.1 Observables of a physical system are represented by hermitean (more precisely self-adjoint) operators on a complex Hilbert space $\mathcal{H}$ **.**

3.2 The eigenvalues of such an operator (which are necessarily real numbers) are the possible outcomes of measuring it.

3.3 States of a physical system are described vectors (more precisely rays) in the Hilbert space.

3.4 If the system is in state $|\psi >\in \mathcal{H}$ **, the probability of obtaining the value** a **during a measurement of the observable** A **is** $\frac{|<a|\psi>|^2}{|<\psi|\psi>|^2}$ **.**

3.4.1 Here, $A|a >= a|a >$ so that $|a >$ is the eigenstate (assumed to be unique) of eigenvalue a .

3.5 There is a self-adjoint operator H **, the *hamiltonian*, which describes the time evolution of a state**

$$i\hbar\frac{\partial}{\partial t}|\psi >= H|\psi > .$$

3.5.1 This is the *Schrödinger* equation.

3.5.2 A particle of mass m and energy E in a potential $V(x)$ has a wave-function that satisfies

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(x)\right]\psi = E\psi.$$

3.5.3 Read any textbook on Quantum Mechanics to refresh your knowledge. I recommend the book *Principles of Quantum Mechanics* by R. Shankar. But a much deeper discussion is in the third volume of the series by Landau and Lifshitz called *Quantum Mechanics*.

Chapter 4

Tunneling and Alpha Decay

4.0.4 If there are $N(t)$ nuclei in a radioactive sample at time t, the number of that decay in some interval dt is $\nu N(t)dt$ for some *decay constant* ν. Thus

$$\frac{dN}{dt} = -\nu N(t) \Rightarrow N(t) = N(0)e^{-\nu t}.$$

4.1 This is the law of exponential decay of radioactivity.

4.1.1 There are small departures from this law for very small and very large times compared to $\frac{1}{\nu}$. This is a subtlety we will ignore for now.

4.1.2 The half-life $T_{\frac{1}{2}}$ of a nucleus is the time it takes for half the sample to decay:

$$e^{-\nu T_{\frac{1}{2}}} = \frac{1}{2}$$

so that

$$T_{\frac{1}{2}} = \frac{\ln 2}{\nu}$$

4.2 Many heavy nuclei decay by breaking up into smaller ones, in a process called *fission.*

4.2.1 The atomic number (number of protons) is conserved in this process $Z = Z_1 + Z_2$ where Z is the atomic number of the original nucleus and Z_1, Z_2 those of the daughter nuclei.

4.2.2 The mass of the daughter nuclei add up to a bit less than the mass of the original nucleus; the remaining mass being converted to the kinetic energy Q of the fragments

$$M = m_1 + m_2 + Q/c^2$$

4.2.3 A very asymmetrical form of fission is when one of the nuclei (say the first) is an alpha particle. $Z_2 >> Z_1 = 2$ in this case. If the recoil of the heavy daughter nucleus is ignored, the alpha particle will have a characteristic energy, determined by the mass of the parent and daughter nuclei. These energy varies from about 4 MeV to about $10MeV$. Often the daughter nucleus undergoes further decays. Many such nuclear decay chains have been identified and tabulated.

4.2.4 This kind of nuclear chemistry is a very useful but ugly subject. The applications are to designing nuclear weapons, nuclear reactors to produce energy,creating radioactive isotopes for use in medicine etc. There are even very mundane applications. For example, Polonium 210 emits alpha particles which are used to counter static electricity built up by paint brushes. It was also used in a notorious case of murder by Polonium poisoning of a former KGB agent.

4.2.5 The figure gives a typical chain of Naturally occurring nuclear decays.

4.2.5.1 Recall that beta decay is the emission of an electron and a pair of neutrinos when a neutron converts itself into a proton. The neutrinos are almost impossible to detect without very specialized equipment. The electric charge or atomic number Z of the nucleus increases by one during beta decay. The number of neutrons plus protons remains the same.

4.2.6 The half-life of the decay (the time it takes for half of a sample of nuclei to have decayed) varies with the energy: the higher the energy of the alpha particle the shorter the half-life.

4.3 The half-life of alpha decay is *extremely* sensitive to the energy of the alpha particle.

4.3.1 As E varies from 4 to 10 MeV, the lifetime varies from a microsecond to 10^{10} years: by a factor of 10^{20} .

4.4 It is found that the logarithm of the half-life is a linear function of $\frac{1}{\sqrt{E}}$

$$\log_{10} T_{\frac{1}{2}} = \frac{B}{\sqrt{E}} + C$$

This is known as the Geiger-Nuttal Law.

4.4.0.1 A good empirical rule is

$$\log_{10} T_{\frac{1}{2}} = \frac{146}{\sqrt{E}} - 53.3$$

where the half-life is measured in seconds and the energy in MeV.

4.4.1 Originally Geiger and Nuttal did not quite get the power of E in the above law correct; this version is a refinement.

4.4.2 The quantities B and C are independent of energy but depend on the parent nucleus. But it turns out that to a good approximation B itself is proportional to the atomic number of the daughter nucleus.

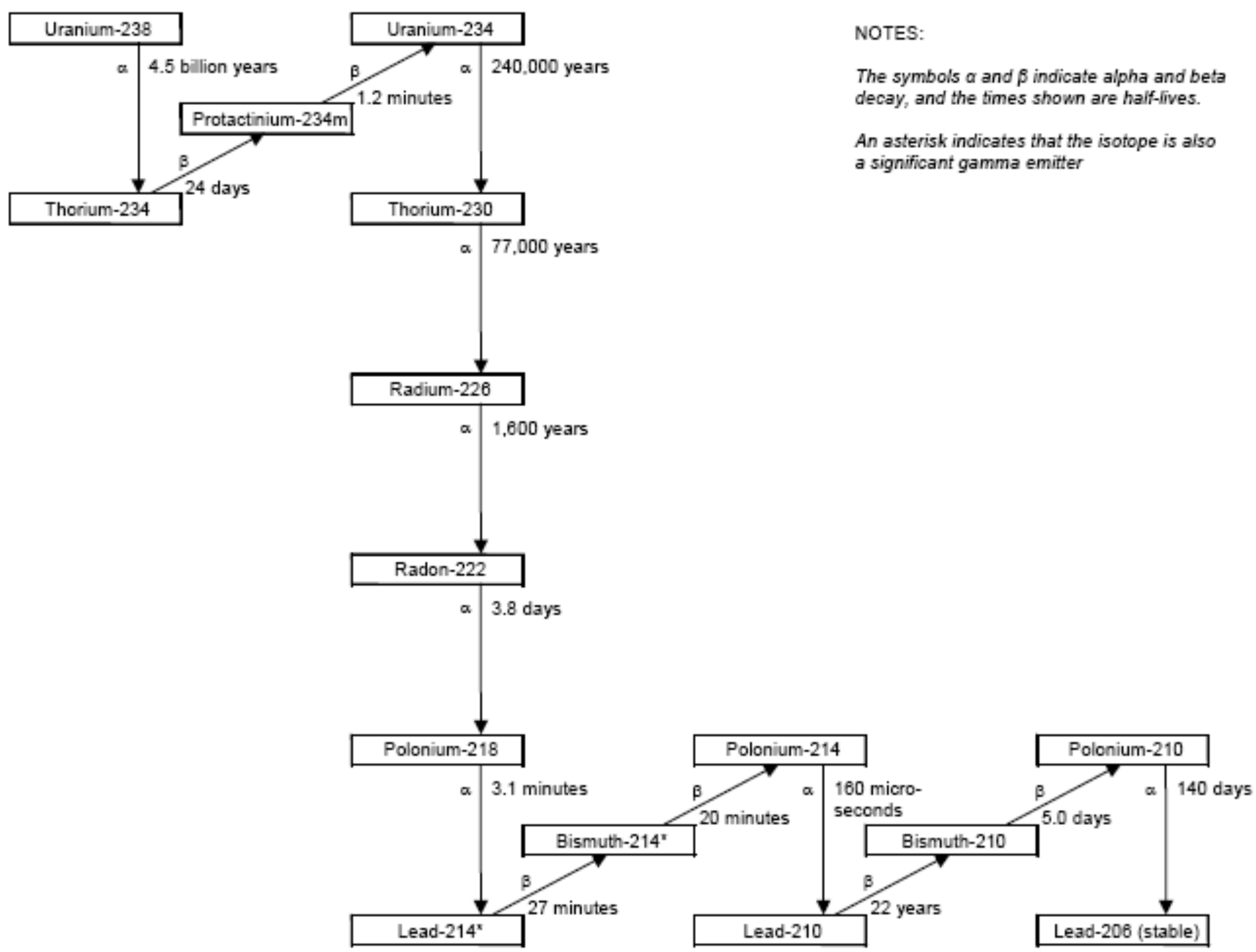

Figure 4.1: Natural Radioactive Decays

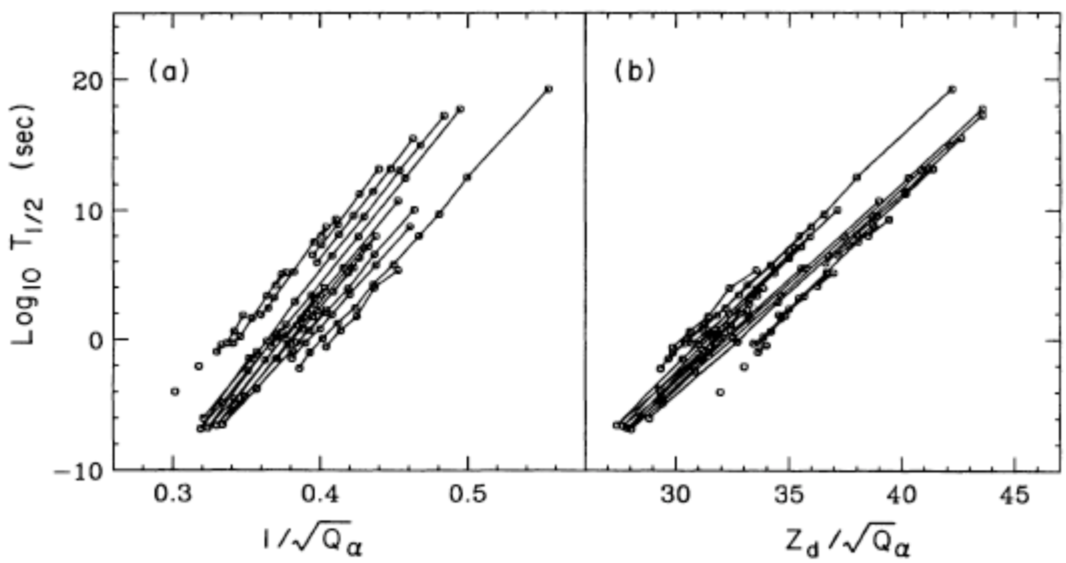

Figure 4.2: Plot of Alpha Halflives ($T_{\frac{1}{2}}$) vs Energy Q_α in MeV. Z_d is the Atomic number of the daughter nucleus. From: B. A. Brown Phys. Rev. C46,811(1992)

4.4.3 In Fig(4.1) what we call E is called Q_α and our Z_2 is called Z_d . This is the traditional notation in Nuclear Physics.

4.5 What explains this extreme sensitivity to energy?

4.5.1 In the early days of quantum mechanics, Gamow (and independently, Gurney and Condon) showed that this is explained by the phenomenon of tunneling.

4.6 One of the important effects of quantum mechanics is tunneling: a particle can escape a potential barrier even if its energy is less than the height of the barrier.

4.6.1 An alpha-particle can escape a heavy nucleus even if it does not seem to have enough energy to go overcome the nuclear force. The energy of the alpha particle is of the order of 10 MeV. The height of the potential barrier it faces is about 25 MeV (see below for a more accurate estimate). Classically, alpha decay is simply not allowed. Quantum mechanically the probability fort it to happen is small; and it depends very sensitively on energy.

4.6.2 In the classically forbidden region the particle has *negative* kinetic energy. Yet, remarkably, quantum mechanics predicts that in this region motion is much like in classical mechanics but with *imaginary time*. Such a particle in classically forbidden motion is nowadays called an *instanton*, following G. 't Hooft who came up with this elegant interpretation of tunneling.

4.7 Let us start with the Schrödinger equation for the wavefunction of a particle of mass m and energy E in a potential $V(x)$:

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(x)\psi = E\psi.$$

4.7.1 We need to understand the situation where quantum effects are small, so that the tunneling probability is small. This is called the semi-classical approximation, where the terms proportional to $\hbar$ are small.

4.7.2 It would be wrong to argue that the first term in the Schrödinger equation is negligible, because the wave-function itself adoes not have a good limit as $\hbar \to 0$.

4.7.3 This can be understood by considering a free particle:

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi$$

A solution is a plane wave

$$\psi = e^{\frac{i}{\hbar}p\cdot x}$$

where p is the momentum; $\frac{p^2}{2m} = E$. In the classical limit $\hbar \to 0$, the physical quantities E and p remain finite but the wavefunction has no limit: it oscillates rapidly and averages out to zero.

4.7.4 To study the limit $\hbar \to 0$ it is useful to make the change of variables

$$\psi = e^{\frac{iW}{\hbar}}$$

in the Schrodinger equation:

$$\frac{(\nabla W)^2}{2m} + V(x) - E + \frac{i\hbar}{2m}\nabla^2 W = 0.$$

4.7.5 The point of this is that W ,unlike ψ has a good limit as $\hbar \to 0$, so that the last term can be ignored because of the factor of $\hbar$. This is the semi-classical approximation:

$$\frac{(\nabla W)^2}{2m} + V(x) = E.$$

4.7.6 In fact ∇W has the physical meaning of momentum in this approximation. For a free particle $W = p \cdot x$ so that $\nabla W = p$. Recall that in quantum mechanics, momentum is represented by the operator $\hat{p} = -i\hbar\nabla$; so $\hat{p}\psi = \nabla W \psi$. If $\psi = e^{\frac{i}{\hbar} p \cdot x}$, we get an eigenstate of momentum. This corresponds to the case where ∇W is a constant. One way to understand the semi-classical approximation is that the momentum is slowly varying, so that ψ can be thought of as a wave whose wavelength is slowly varying.

4.7.7 In the classically allowed region $E > V(x)$ and $(\nabla W)^2 > 0$; otherwise ∇W is imaginary. Thus, in the classically allowed region W is real and the wavefunction is oscillatory, while in the classically forbidden region it decays exponentially.

4.7.8 In a system with one degree of freedom, suppose that the regions $x < x_1$ and $x > x_2$ are classically allowed and that that $x_1 < x < x_2$ is forbidden. That is, $V(x) > E$ in the forbidden region. Then we can solve the above equation and get the wavefunction in the forbidden region as

$$\psi(x) \approx e^{-\frac{\surd(2m)}{\hbar}\int_{x_1}^{x} \surd[E - V(x)]dx}$$

Once a particle shows up at x_1 , he probability of tunneling across this barrier to x_2 is the square of the wavefunction at x_1 :

$$e^{-2\frac{\surd(2m)}{\hbar}\int_{x_1}^{x_2} \surd[V(x) - E]dx}$$

4.7.9 As an application consider an alpha-particle inside a heavy nucleus. Outside the nucleus (of radius r_0) the energy of the alpha particle is

$$V(r) = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0}\frac{1}{r}, \quad \text{for } r > r_0$$

with $Z_1 = 2$. Also Z_2 is the atomic number of the nucleus *after* the decay. The potential inside the nucleus is a complicated superposition of interactions many neutrons and protons. It turns out to be a good approximation to think of it as a constant

$$V(r) = V_0 \quad \text{for } r < r_0.$$

Even the simple choice $V_0 = 0$ gives a good approximation.

4.7.10 r_0 is the sum of the radii of the alpha-particle and the daughter nucleus: that is the minimum distance before the alpha is trapped in the nucleus.

4.8 All nuclei have about the same density.

4.8.1 Thus the radius of a nucleus is proportional to the cuberoot of the number of particles in it. In fact $r \approx 1.2(A)^{\frac{1}{3}}$ Fermi where A is the number of neutrons plus protons. For the case of alpha decay this gives $r_0 = 1.2[A_2^{\frac{1}{3}} + 4^{\frac{1}{3}}] \approx 9$ fm since the typical value of A_2 is about 225.

4.8.2 The height of the barrier is

$$V(r_0) = Z_1 Z_2 \frac{e^2}{4\pi\epsilon_0} \frac{1}{r_0} = Z_1 Z_2 \frac{e^2}{4\pi\epsilon_0 \hbar c} \frac{\hbar c}{r_0} = Z_1 Z_2 \alpha \frac{\hbar c}{r_0}$$

4.8.2.1 The dimensionless constant

$$\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} \approx \frac{1}{137}$$

appears often in physics. It is called the *fine structure constant*. It is a coincidence that it is called α : it doesn't have anything directly to do with alpha decay. It appears whenever we study an electromagnetic quantum phenomenon, being the dimensionless combination of $e, \hbar$ and c .

4.8.2.2 The fundamental constant $\hbar c = 197\text{Mev fm}$. This is a useful number to remember in making estimates such as these.

4.8.2.3 Thus the height of the barrier is about $V(r_0) \approx 28$ MeV.

4.9 We can calculate half-life of the nucleus emitting the alpha particles as a function of their energy E.

4.9.1 The distance a at which the alpha particle will emerge is given by

$$\frac{Z_1 Z_2 e^2}{4\pi\epsilon_0}\frac{1}{a} = E \Rightarrow a = Z_1 Z_2 \alpha \frac{\hbar c}{E}.$$

Hence the probability of tunneling is

$$\Gamma := e^{-\gamma} = e^{-\frac{2\sqrt{2mE}}{\hbar}\int_{r_0}^{a}\sqrt{\frac{a}{r}-1}dr} = e^{-\frac{2\sqrt{2mE}a}{\hbar}\int_{\frac{r_0}{a}}^{1}\sqrt{\frac{1}{\rho}-1}d\rho}$$

4.9.1.1

$$\int_{\frac{r_0}{a}}^{1}\sqrt{\frac{1}{\rho}-1}d\rho = \int_0^1\sqrt{\frac{1}{\rho}-1}d\rho - \int_0^{\frac{r_0}{a}}\sqrt{\frac{1}{\rho}-1}d\rho \tag{4.1}$$

$$= \frac{\pi}{2} - \int_0^{\frac{r_0}{a}}[1-\rho]^{\frac{1}{2}}\frac{d\rho}{\sqrt{\rho}} \tag{4.2}$$

$$= \frac{\pi}{2} - \int_0^{\frac{r_0}{a}}\left[1-\frac{1}{2}\rho+\cdots\right]\frac{d\rho}{\sqrt{\rho}} \tag{4.3}$$

$$= \frac{\pi}{2} - 2\left[\frac{r_0}{a}\right]^{\frac{1}{2}} + \cdots$$

4.9.2 Thus recalling the definition of a ,

$$\sqrt{E}a = Z_1 Z_2 \alpha \frac{\hbar c}{\sqrt{E}}, \quad \sqrt{Ea} = \sqrt{Z_1 Z_2 \alpha \hbar c}$$

Thus,

$$\gamma \approx \frac{2\sqrt{2mE}a}{\hbar}\left[\frac{\pi}{2} - 2\sqrt{\frac{r_0}{a}} + \cdots\right] \tag{4.4}$$

$$= 2\frac{\sqrt{2m}}{\hbar}\left[\frac{\pi}{2}Z_1 Z_2 \alpha \frac{\hbar c}{\sqrt{E}} - 2\sqrt{Z_1 Z_2 \alpha \hbar c r_0} + \cdots\right] \tag{4.5}$$

$$= \pi Z_1 Z_2 \alpha\sqrt{\frac{2mc^2}{E}} - 4\sqrt{Z_1 Z_2 \alpha}\sqrt{\frac{2mc^2 r_0}{\hbar c}} + \cdots$$

Thus the leading term depends on energy as $\frac{1}{\sqrt{E}}$ and the next term is independent of energy. The first term we ignored (repesented by $\cdots$) varies as $\sqrt{E}$, the one after that as E and so on.

4.10 The probability of tunneling $\Gamma = e^{-\gamma}$ determines the rate of escape per unit time of the alpha particle. The half-life varies inversely as this rate.

4.10.1 Each time an alpha particle reaches the edge of the nucleus it has this probability to tunnel out. The expected rate of escape per unit time, ν, will be this probability Γ times the number of times it hits the nuclear wall in unit time. The latter is given by the velocity divided by the diameter of the nucleus. So,

$$\nu \approx \frac{\sqrt{\frac{2E}{m}}}{2r_0} e^{-\gamma}$$

4.10.2 Putting the pieces of the above argument together

$$T_{\frac{1}{2}} = 2\ln 2 \frac{r_0}{c}\sqrt{\frac{mc^2}{2E}} e^{\gamma}.$$

4.10.3 Because of the tremendous range in the values of the half-life, it is more convenient to think in terms of the logarithm of $T_{\frac{1}{2}}$. It is conventional to take the logarithm to base 10 (instead of e as is usual in higher mathematics) and to express $T_{\frac{1}{2}}$ in seconds.

$$\ln T_{\frac{1}{2}} = \gamma - \frac{1}{2}\ln\frac{2E}{mc^2} + \ln\frac{r_0}{c} + \ln[2\ln 2].$$

$$\log_{10} T_{\frac{1}{2}} = \frac{1}{\ln 10}\gamma - \frac{1}{2}\log_{10}\frac{2E}{mc^2} + \log_{10}\frac{r_0}{c} + \log_{10}[2\ln 2].$$

4.10.4 Looking up the formula for γ we get

$$\begin{aligned} \log_{10} T_{\frac{1}{2}} &= \frac{\pi}{\ln 10} Z_1 Z_2 \alpha \sqrt{\frac{2mc^2}{E}} - \frac{4}{\ln 10}\sqrt{Z_1 Z_2 \alpha}\sqrt{\frac{2mc^2 r_0}{\hbar c}} + \\ &\quad \log_{10}\frac{r_0}{c} + \log_{10}[2\ln 2] - \frac{1}{2}\log_{10}\frac{2E}{mc^2} + \cdots \end{aligned} \tag{4.6}$$

This is of the form

$$\log_{10} T_{\frac{1}{2}} = \frac{B}{\sqrt{E}} + C_0 + C_1 \log E + C_2\sqrt{E} + C_3 E + \cdots$$

with

$$B = \frac{\pi}{\ln 10} Z_1 Z_2 \alpha \sqrt{2mc^2}$$

When $m = 4 \times 940$ MeV, $Z_1 = 2, Z_2 = 90$ this gives a numerical value

$$B = 2\pi \times 90 \times \frac{1}{137}\sqrt{2 * 4 * 940}\frac{1}{\ln 10} = 155$$

in remarkable agreement with the constant $B = 146$ in the experimental Geiger-Nuttal law.

4.10.5 B does depend somewhat on the nucleus, through Z_2. That could explain the remain discrepancy, as the experimental value is an average over many decays. Also we have ignored many other effects such as the potential energy of the α particle inside the nucleus. Note that in the figure the fit is better when this dependence is taken into account. What we call Z_2 is called Z_d in the paper from which the diagram is taken. Also our E is called Q_α. That is the jargon of Nuclear Physics.

4.10.6 Thus we recover the Geiger-Nuttal law with the first two terms. Moreover, the constant first term B is proportional to Z_2 , the atomic number of the daughter nucleus. The first two terms are mostly what matter. The constant C in the Geiger-Nuttal law is an average over the range of energies E of the slowly varying function $C_0 + C_1\sqrt{E} + \cdots$.

Chapter 5

Semi-Classical Mechanics of Tunneling

5.1 Quantum Mechanics is well-approximated by classical mechanics when the wavelength of the system is small compared to its size.

5.1.1 The derivation of classical mechanics as an approximation to quantum mechanics is as outlined in the last chapter. We start with the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(x)\psi = E\psi.$$

Because the wavefunction does not have a limit as $\hbar \to zero$ we change variables to $\psi(x) = e^{\frac{i}{\hbar}W(x)}$. The quantity W has a limit as $\hbar \to 0$. It has the classical meaning of an 'eikonal'. Its gradient is the momentum.

5.1.2

$$\frac{(\nabla W)^2}{2m} + V(x) + \frac{i\hbar}{2m}\nabla^2 W = E$$

In the limit $\hbar \to 0$

$$\frac{(\nabla W)^2}{2m} + V(x) = E$$

Since there is no $\hbar$ in the story anymore, this is a statement within classical mechanics. In fact it was discovered long before quantum mechanics: it is known as the Hamilton-Jacobi equation of classical mechanics.

5.1.3 Any first order PDE is equivalent to a system of Ordinary differential equations. The solutions to these ODEs determine curves which are the *characteristics* of the PDE. In our case, these ODE are the equations for the derivative of position and momentum:

$$\frac{p^2}{2m} + V(x) = E \Rightarrow$$

$$p = m\frac{dx}{dt}, \quad \frac{dp}{dt} = -\nabla V.$$

There is a systematic way of deriving them using *Poisson brackets*. You can read my notes on classical mechanics for example. Or Goldstein's or Landau's books.

5.1.4 Classical mechanics follows when the solution for W is real. But when $E < V(x)$ there is still a solution with purely imaginary W . It describes tunneling. We saw that this is a real and important physical effect. Is there a way of determining the path of the particle in the forbidden region?

5.1.5 Mathematically, we get the same kind of problem as above if we put $\tilde{W} = iW$, $\tilde{p} = ip, t = i\tilde{t}$ in the forbidden region:

$$-\frac{(\nabla W)^2}{2m} + V(x) = E$$

$$\frac{d\tilde{p}}{d\tilde{t}} = \nabla V, \quad m\frac{dx}{d\tilde{t}} = \tilde{p}.$$

The solutions to these ODEs are the characteristic curves of the above first order PDE.

5.1.6 Another equivalent point of view is that the Newton's equations we replace $t = i\tilde{t}$:

$$-m\frac{d^2x}{d\tilde{t}^2} = -\nabla V.$$

In other words it is as if we reversed the sign of the force.

5.1.7 Thus there are two 'sheets' for the classical phase space: one with real time and another with imaginary time. These two worlds touch each other at the turning points, where the momentum vanish. For a system with one degree of freedom, the phase space is two copies of R^2 glued along the x -axis, which is where the momentum vanishes. This is not a manifold. A classical path that intersects the x -axis can continue either by turning around, or by continuing into the other phase plane. On the tunneling phase plane the path will be parametrized by $\tilde{t}$ and will continue until it hits a point on the x -axis again, another turning point. Again, it can either turn around or pass into the classically allowed phase plane.

5.2 In general W is a complex-valued function, not real or purely imaginary.

5.2.1 C. P. Burgess, American Journal of Physics – November 1991 – Volume 59, Issue 11, pp. 994-998; also available online.

5.2.2 Thus in general, momentum is a complex-valued vector during tunneling. It may have some real components and some imaginary components. Therefore in a general basis, its components will be complex,with both a real part and an imaginary part.

5.2.3 The tunneling phenomenon has an optical analogue. Total internal reflection (analogous to a turning point of classical motion) occurs when a light ray passes from a region with large refractive index to one of small index, with a small enough incident angle. If, after passing through a region of low refractive index, the light can emerge into another region of high index, some of the light will tunnel through. This wave that is forbidden in the ray approximation is called an evanescent wave. Its wave-number has an imaginary normal component but real tangential components. What is remarkable is that the evanescent wave of light follows a definite path even in the forbidden region.

Chapter 6

Rotations

6.1 The distance between two points with co-ordinates $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ is given by $\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$.

6.2 If we translate both vectors by the same amount $x \to x + a$ and $y \to y + a$, the distance is unchanged.

6.3 Similarly if we rotate both the same way, the distance is unchanged.

6.4 The length of a vector is $\sqrt{x^T x}$ where x is thought of as a 1×3 matrix; i.e., a column vector.

6.4.1 x^T stands for the transpose, which is a 1×3 'matrix'; i.e., a row vector.

6.5 A rotation is described by a 3×3 matrix R :

$$x \to Rx.$$

To preserve the length $(Rx)^T(Rx) = x^T x$; i.e., $x^T(R^T R)x = x^T x$.

6.6 Thus a rotation must satisfy

$$R^T R = 1.$$

A matrix satisfying such a condition is called an *orthogonal matrix.*

6.6.1 Not all orthogonal matrices describe rotations. For example $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is orthogonal; it reflects the first component while leaving the others unchanged. This is not possible by any rotation. On the other hand $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ reflects the first and second components; this just a rotation around the third axis by 180^o .

6.7 A rotation is an orthogonal matrix whose determinant is one.

6.7.1 Since $\det AB = \det A \det B$ and $\det A^T = \det A$, we can deduce that for and orthogonal matrix $(\det R)^2 = 1$. Thus $\det R = \pm 1$. Under small changes of the matrix elements of R , the determinant cannot change: it would have to jump from 1 to -1 if it were to change. Since all rotations can be got from the identity by a continuous change of matrix elements (change the angle of rotation), they have to have determinant one. An orthogonal matrix with determinant -1 is a combination of a rotation and a reflection.

6.8 The set of orthogonal matrices is a *group*

6.8.1 The product of two orthogonal matrices is orthogonal, the identity is an orthogonal matrix and the inverse of an orthogonal matrix is one as well. Moreover the multiplication of matrices is associative.

6.8.2 This group is denoted by $O(3)$

6.9 The set of Special Orthogonal matrices, $SO(3)$, which represent rotations, is a subgroup.

6.10 An infinitesimal transformation $R = 1 + A$ is orthogonal if $A^T + A = 0$; i.e., infinitesimal rotations are described by anti-symmetric matrices.

6.11 An arbitrary anti-symmetric matrix can be written as a linear combination of the basic ones

$$S_{12} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad S_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

6.12 A *Lie algebra* is a linear vector space along with a bilinear operation satisfying

$$[A, A] = 0, \quad [A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0.$$

6.13 The *commutator* of two matrices is defined to be $[A, B] = AB - BA$.

6.14 The set of anti-symmetric matrices form a Lie algebra, using the commutator.

6.14.1 The commutation relations of the basis elements describe this algebra completely:

$$[S_{12}, S_{13}] = -S_{23}, \quad [S_{12}, S_{23}] = S_{13}, \quad [S_{13}, S_{23}] = -S_{12}$$

6.14.2 It is special to the case $n = 3$ that $SO(n)$ has dimension n ; in general the dimension is $\frac{n(n-1)}{2}$. It is convenient to take advantage of this coincidence and use a simplified notation

$$S_3 = S_{12}, S_1 = S_{23}, \quad S_2 = -S_{13}$$

which satisfy

$$[S_3, S_2] = S_1, \quad [S_3, S_1] = -S_2, \quad [S_1, S_2] = S_3$$

6.15 What is the Lie algebra of rotations in n dimensions?

Chapter 7

Lorentz Invariance

7.1 It is an astonishing physical fact that the speed of light (in vacuum) is the same in all reference frames.

7.1.1 This is not true of other waves; for example the speed of sound is different for observers moving at different velocities with respect to air.

7.2 The law of addition of velocities has to be modified to be consistent with this peculiar result:

$$v_1 \oplus v_2 = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}.$$

7.2.1 If we require that the addition of velocities forms a group, and that c added to any velocity give c, this is the only answer.

7.2.2 The easiest way to check the group property to make the change of variable

$$\frac{v}{c} = \tanh\theta$$

and use the addition formula for $\tanh$.

7.3 This means that the shape of the wavefront of light is the same for all observers.

7.3.1 If a pulse of light is emitted at the origin at time $t = 0$, it will spread along the cone

$$c^2t^2 - x_1^2 - x_2^2 - x_3^2 = 0, t > 0.$$

7.4 The laws of physics have to the same for all observers moving at constant velocity relative to each other.

7.4.1 This innocuous statement has important consequences when combined with the fact the the velocity of light is the same for all such observers.

7.4.2 The Newtonian notions of time, space, energy, momentum all need to be modified.

7.4.3 Minkowski noted that the addition law for velocities has a simple geometric interpretation: the rule for the distance between two points in space-time is $\sqrt{c^2(t-t')^2 - (x_1 - x_1')^2 - (x_2 - x_2')^2 - (x_3 - x_3')^2}$

7.4.4 If velocity is identified as $\frac{dx}{dt}$, the rule for addition of velocities corresponds to 'rotations' around an imaginary angle. A more precise version is,

7.5 The *Minkowski* inner product of a pair of vectors in space-time is

$$(u, v) = u_0v_0 - u_1v_1 - u_2v_2 - u_3v_3.$$

7.5.1 We can also write this in matrix notation:

$$(u, v) = u^T\eta v \quad \eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

7.6 A *Lorentz transformation* is a linear transformation leaving this inner product invariant:

$$[\Lambda u]^T \eta [\Lambda v] = u^T \eta v, \quad \Rightarrow \Lambda^T \eta \Lambda = \eta.$$

7.6.1 If Λ_1 and Λ_2 satisfy this equation, so will the product and inverse: the set of Lorentz transformations is a group.

7.6.2 $SO(3)$ is a subgroup:

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}, \quad R^T R = 1.$$

7.6.3 By taking determinants it is easy to see that the determinant of a Lorentz matrix is either ± 1 . The sign of the determinant is constant under continuous changes of matrix elements.

7.6.4 The 00 component of the condition says

$$\Lambda_{00}^2 - \Lambda_{01}^2 - \Lambda_{02}^2 - \Lambda_{03}^2 = 1.$$

This is the equation for a hyperboloid of two sheets. So the sign of Λ_{00} is constant under continuous changes of the matrix elements

7.6.5 The set of Lorentz transformations splits into four connected components labelled by the signs of $\det \Lambda$ and Λ_{00} .

7.6.6 The component containing the identity (i.e., $\det \Lambda = 1$ and $\Lambda_{00} > 0$) is the group of *proper Lorentz transformations*. They preserve the orientation of time and do not contain reflection of an odd number of spatial directions.

7.7 Only the proper Lorentz transformations are symmetries of physical laws.

7.7.1 Weak interactions are not invariant under violate parity as well as time reversal.

7.8 The relation of momentum $\mathbf{p}$ to energy E is

$$(p,p)=m^2c^4, \quad p=(E,c\mathbf{p}).$$

7.8.1 When $m>0$ this is a two-sheeted hyperboloid; if $m=0$ this is a cone.

7.8.2 The case $m^2<0$ is un-physical since the sign of energy is not invariant under proper Lorentz transformations. The hyperboloid in this case has a single sheet and contains energies that are negative: some observers will see that the energy of a given vector is is positive and others will see it as negative, so there is no way to exclude negative energies. But then we can make energies as negative as we want, and the system is unstable against emission of arbitrarily large amounts of energy. These non-existent particles are called *tachyons*.

Chapter 8

Relativistic Mechanics

8.1 The best reference for this material is the book *Electrodynamics and Classical Theory of Fields and Particles* **by A. O. Barut (Dover).**

8.2 The path of a massive particle in space-time is a time-like curve; for a massless particle it is a null curve.

8.2.1 The tangent vector at any point is the four-velocity.

8.2.2 In relativistic mechanics, momentum is a four-vector whose components are $(K, c\mathbf{p})$ and time is the zeroth component of the position vector in space-time $(ct, \mathbf{x})$.

8.2.3 It is no longer correct to think of momentum as

$$m\frac{d\mathbf{x}}{dt}$$

8.2.4 It is better to measure time measured by an observer attached to the particle,*proper time* instead of an outside observer. This will be a property of the curve followed by the particle. An infinitesimal change in the proper-time is the length of the tangent vector:

$$d\tau = \sqrt{c^2(dx^0)^2 - (d\mathbf{x})^2}$$

This is a Lorentz scalar, an invariant property of the curve.

8.2.5 A curve is said to be null if every tangent vector is of zero length in the Minkowski metric. Thus $d\tau = 0$. The following discussion does not apply to such particles.

8.2.6 Consider the case of a time-like curve, for which every tangent vector is time-like. The four-velocity

$$v^\mu = \frac{dx^\mu}{d\tau}$$

is then of unit length .

8.2.7 The four-momentum of a particle is

$$p^\mu = mv^\mu.$$

It satisfies

$$p^\mu p^\nu \eta_{\mu\nu} = m^2 c^4.$$

This is the relativistic relation between energy and momentum of a free particle,

8.3 A particle moves along a straightline in the absence of forces.

8.3.1 This is Newton's first law and continues to hold in relativity.

8.3.2 The most interesting forces in relativistic mechanics are usually electromagnetic. Gravitational forces requires general relativity, a much more sophisticated theory. Nuclear and weak forces will be studied later and need elements of the quantum theory.

8.3.3 Recall the equation of motion of a charged particle in non-relativistic mechanics:

$$\frac{d\mathbf{p}}{dt} = e\mathbf{E} + e\frac{\mathbf{v}}{c} \times \mathbf{B}.$$

where $\mathbf{b} = m\mathbf{v}$ is the momentum.

8.3.4 We can derive from it also an equation for the rate of kinetic energy $K = \frac{1}{2}m\mathbf{v}^2$:

$$\frac{dK}{dt} = e\mathbf{v} \cdot \mathbf{E}$$

8.3.5 The magnetic field does no work on the particle. Its effect on a charged particle is to change its direction of motion without changing its energy.

8.4 It will be convenient from now on to set units such that $c = 1$.

8.5 The equation of motion of a charged particle in relativistic motion is

$$\frac{dp_\mu}{d\tau} = eF_{\mu\nu}v_\nu, \quad p_\mu = m\eta_{\mu\nu}\frac{dx^\nu}{d\tau}.$$

This is called the Lorentz equation of motion.

8.5.1

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

8.6 An accelerated particle emits radiation. The loss of energy and momentum due to this radiation in turn has an effect on the motion of the particle. This is called radiation reaction.

8.6.1 Radiation reaction is a small effect in most situations. But in particle accelerators, where charged particles are moving relativistic speeds in a circle, the loss of energy is substantial. To reduce this effect, the accelerators have very large diameter (of the order of 100 km). Also microwaves are used to replenish the lost energy.

8.6.2 This synchrotron radiation can be used as a strong source of X-rays for solid state physics and biological physics experiments. This is what old particle accelerators do when they are past their glory days in high energy physics.

8.7 An old result of Larmor gives the power radiated by a non-relativistic charged particle:

$$P = \frac{2}{3}\frac{e^2}{c^3}\mathbf{a}^2$$

where $\mathbf{a}$ is the acceleration.

8.7.1 This is in cgs units. In SI units there will be $\frac{e^2}{4\pi\epsilon_0}$ in place of e^2 .

8.7.2 Thus a charged particle moving in a circular trajectory will lose energy. If v is the velocity and r the radius of the orbit, $a = \frac{v^2}{r} = \frac{2K}{mr}$ where K is the kinetic energy.

$$P = \frac{8}{3}\frac{e^2}{m^2c^3}\frac{K^2}{r^2}$$

Thus, for a given energy, the radius of the circle has to be large to limit the loss of energy due to radiation.

8.7.3 Another way of looking at it is to ask for the rate of loss of energy of a particle moving in a circular orbit in a magnetic field. The angular velocity is $\omega = \frac{eB}{mc}$ and the acceleration $a = \omega^2 r = \omega v = \omega\sqrt{\frac{2K}{m}}$ so that so that

$$\frac{dK}{dt} = -\frac{4}{3}\frac{e^2}{mc^3}\omega^2 K$$

Thus the energy loss is exponential.

8.8 The reaction force due to radiation for a non-relativistic particle is

$$\mathbf{f} = \frac{2}{3}\frac{e^2}{c^3}\frac{d^2\mathbf{v}}{dt^2}$$

8.8.1 That is, the force is proportional to the time derivative of acceleration.

8.9 In the relativistic theory, Dirac showed that the four-vector describing the force due to radiation reaction is

$$f^\mu = \frac{2}{3}e^2\left[\eta^{\mu\nu} - v^\mu v^\nu\right]\frac{d^2 v_\nu}{d\tau^2}.$$

Note that the tensor in front just projects out the part of the vector $\frac{d^2 v_\nu}{d\tau^2}$ **that is orthogonal to** v^μ **.**

8.9.1 But if we directly put this into the equation of motion

$$m\frac{dv_\mu}{d\tau} = eF_{\mu\nu}v^\nu + f_\mu$$

we will get a *third order* differential equation. What is worse, this equation has 'runaway' solutions with energy that grows with time. It is possible to choose initial conditions such that this does not happen, but then the particle will start accelerate *before* we exert a force on it! Total confusion reigned in this subject until recently.

8.9.2 A careful analysis by Spohn and by Rohrlich has resolved the problem recently. H. Spohn Europhys. Lett. 50,287(2000); F. Rohrlich Phys. Lett. A 283,276 (2001).

8.9.3 In brief, the idea is to treat the radiation reaction as a small perturbation. This means that we compute the force f_μ by putting into it it the time derivative of acceleration as determined by the usual Lorentz force.

$$f_\mu = \frac{2}{3}\frac{e^3}{m}\left[\eta^{\mu\nu} - v^\mu v^\nu\right]\left[\nabla_\sigma F^{\nu\rho}\ v^\sigma v_\rho + \frac{e}{m}F^{\nu\rho}F_{\rho\sigma}v^\sigma\right]$$

Thus the full equation is still a second order ordinary differential equation, although non-linear:

$$\frac{dv_\mu}{d\tau} = \frac{e}{m}F_{\mu\nu}v^\nu + \frac{2}{3}\frac{e^3}{m^2}\left[\eta^{\mu\nu} - v^\mu v^\nu\right]\left[\nabla_\sigma F^{\nu\rho}\ v^\sigma v_\rho + \frac{e}{m}F^{\nu\rho}F_{\rho\sigma}v^\sigma\right]$$

Remarkably, this equation was given without much comment, in the textbook by Landau and Lifshitz, *Classical Theory of Fields* many years before the work of Spohn and Rohrlich.

8.9.4 Exact solutions for a particle in constant electric and magnetic fields have been found. The exact solution for a Coulomb field is not known yet.

8.10 Conservation of momentum and energy provide useful constraints on the scattering of relativistic particles.

8.10.1 This example is adapted from Perkins *Introduction to High Energy Physics*. Suppose a particle of mass m , energy E , momentum $\mathbf{p}$ collides with a stationary target of mass M producing a new particle of mass M' . Conservation of energy and momentum give, with $p = (E, \mathbf{p}), P = (M, \mathbf{0})$,

$$p + P = P' \Rightarrow p^2 + P^2 + 2p \cdot P = M'^2, \quad m^2 + M^2 + 2EM = M'^2.$$

Thus this can happen only when the incident particle has energy

$$E = \frac{M'^2 - m^2 - M^2}{2M}.$$

Also, the particle is produced with momentum $\mathbf{p}$; in particular it is moving in the same direction as the incoming particle.

Chapter 9

Symmetries and Conservation Laws in Quantum Mechanics

9.1 A symmetry is a linear transformation on states that (i) preserves the probabilities of transitions between states and (ii) commutes with the hamiltonian (i.e., preserves the energy of states).

9.1.1 Recall that the probability of finding a particle in state ψ in another state ϕ is $| < \phi|\psi > |$ (assuming that $|\psi| = |\phi| = 1$.) If the symmetry is represented by a linear transformation it must satisfy

$$| < L\phi|L\psi > | = 1.$$

Also recall the definition of the hermitean conjugate (adjoint)

$$< L\phi|\psi > = < \phi|L^\dagger\psi >$$

Thus we see that one way to satisfy the condition is to have

$$L^\dagger L = 1$$

That is, unitary transformations. Most symmetries are of this type. There are some slight generalizations of unitary transformations allowed as well (”projective representations”) but we will not need them in this course.

9.1.2 If a state has energy E

$$H\psi = E\psi$$

a symmetry must take it to another state of the same energy:

$$H(L\psi) = E(L\psi).$$

This is satisfied if

$$HL = LH.$$

That is, if the hamiltonian commutes with the symmetry operator.

9.2 A symmetry is represented by a unitary operator that commutes with the hamiltonian:

$$L^\dagger L = 1, \quad HL - LH = 0.$$

9.2.1 An exceptional case is time reversal, which is an anti-linear operator

$$\Theta(a\psi + b\phi) = a^*\Theta\psi + b^*\Theta\phi.$$

We won't have much more to say about this case for now; we will only consider the case of linear operators here.

9.2.2 An example is Parity, which reverses the sign of the co-ordinates of a particle

$$P\psi(\mathbf{x}) = \psi(-\mathbf{x}).$$

Clearly $P^2 = 1$. The Schrödinger equation for a free particle is invariant under this transformation

$$-\frac{\hbar^2}{2m}\nabla^2\psi = -i\hbar\frac{\partial\psi}{\partial t}.$$

Another way of seeing that this is a symmetry is that the operator P commutes with the hamiltonian

$$H = -\frac{\hbar^2}{2m}\nabla^2, \quad PH = HP.$$

Thus if ψ is a state with energy E ,

$$H\psi = E\psi$$

so will be $P\psi$.

9.2.3 Even with a potential parity continues to be a symmetry if

$$V(-\mathbf{x}) = V(\mathbf{x}).$$

9.2.4 For example consider a particle in one dimension with a potential

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V, \quad V(x) = \lambda(x^2 - a^2)^2, \quad \lambda > 0.$$

There are two minima at $x = \pm a$. The eigenstates of energy can also be simultaneously eigenstates of parity because $[H, P] = 0$. It turns out that the ground state is of even parity

$$\psi(-x) = \psi(x)$$

while the first excited state is of odd parity

$$\psi(-x) = -\psi(x).$$

9.3 Translation invariance leads to conservation of momentum.

9.3.1 The translation by a is represented by the operator

$$T(a)\psi(x) = \psi(x + a).$$

A free particle on a line has hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V$$

with a constant potential. Thus whether we apply the hamiltonian before or after a translation we get the same effect on a wavefunction:

$$HT(a) = T(a)H.$$

9.3.2 For a particle moving in one dimension, an infinitesimal translation is represented by the derivative operator:

$$\psi(x+a) \approx \psi(x) + a\frac{\partial\psi}{\partial x} + \cdots$$

Thus if a system is invariant under translation, its hamiltonian must satisfy

$$[H, \frac{\partial}{\partial x}] = 0.$$

The operator $\frac{\partial}{\partial x}$ is anti-hermitean. The corresponding hermitean operator is $-i\frac{\partial}{\partial x}$. If we multiply by $\hbar$ we get the momentum operator

$$p = -i\hbar\frac{\partial}{\partial x}.$$

Thus translation invariance implies the conservation of the momentum:

$$[H, p] = 0.$$

Similar arguments apply to each component of momentum of a free particle moving in R^3 .

9.4 Rotation invariance implies conservation of angular momentum. The infinitesimal generators of rotation are

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \quad \mathbf{p} = -i\hbar\frac{\partial}{\partial \mathbf{r}}.$$

9.5 They satisfy the relations

$$[L_1, L_2] = i\hbar L_3, \quad [L_2, L_3] = i\hbar L_1, \quad [L_3, L_1] = i\hbar L_2.$$

9.5.1 A good reference is R. Shankar *Principles of Quantum Mechanics* Chapter 14.

9.6 In quantum mechanics, a particle can have angular momentum even when its momentum is zero.

9.7 Total angular momentum is the sum of the orbital angular momentum and an intrinsic angular momentum

$$\mathbf{J} = \mathbf{L} + \mathbf{S}.$$

The components of $\mathbf{S}$ are a set of three matrices satisfying

$$[S_1, S_2] = i\hbar S_3, \quad [S_2, S_3] = i\hbar S_1, \quad [S_3, S_1] = i\hbar S_2.$$

9.7.1 The simplest choice is $S = 0$. There are several such spin zero particles; e.g., the alpha particle.

9.7.2 The next simplest choice is

$$S_1 = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad S_2 = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad S_3 = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These are 'spin half' particles, since the maximum eigenvalue of a component of spin is half of $\hbar$. An electron, a proton, a neutron are all examples of such particles.

9.7.3 The photon has spin one. But it cannot be described by the above theory because it moves at the speed of light. We need relativistic quantum mechanics for that.

9.7.4 There are a set of particles called Δ that have spin $\frac{3}{2}$. Their spin is represented by 4×4 matrices. There are particles with even higher spin but they tend to be unstable.

Chapter 10

The Neutron and Isotopes

10.1 It was found that the nucleus contains an electrically neutral particle in addition to the proton, which has charge one.

10.1.1 While the proton has a mass of $m_p = 938$ MeV, the neutron has mass $m_n = 939.5$ Mev, which is astonishingly close to be a coincidence.

10.1.2 During beta decay, a neutron converts itself to a proton and an electron (and an anti-neutrino which is often hard to detect).

10.1.3 The atomic number of a nucleus is the number of protons in it. Its atomic mass number is the number of protons plus the number of neutrons.

$$A = N + Z.$$

10.2 Nuclei with the same number of protons but differing numbers of neutrons will form atoms with almost identical chemical properties.

10.3 This explains the existence of *isotopes* which are atoms with identical chemistry but different masses.

10.3.1 For example, 99.8% of oxygen in nature is the isotope with atomic mass number 16 . But oxygen also has stable isotopes of masses 17 and 18 and several unstable ones.

10.3.2 Hydrogen has the simplest nucleus with just a single proton. It has a stable isotope of atomic mass number 2 (deuterium) and an unstable isotope (tritium) with atomic mass 3 and a half life of 12.32 years.

10.3.3 The abundant isotope of Helium has $Z = 2, A = 4$. Its nucleus is the alpha particle. Another stable isotope is He_3 which is the product of tritium decay.

10.4 The binding energy B of a nucleus containing N neutrons and Z protons is

$$B(N, Z) = [M(N, Z) - Nm_n - Zm_p]c^2$$

where $M(N, Z)$ is the mass of the nucleus.

10.4.1 The binding energy of deuterium is 2.2 Mev.

10.4.2 The condition for stability of a nucleus against beta decay is that

$$M(N - 1, Z + 1) - M(N, Z) < m_n + m_e \approx 938.58 \text{ Mev} \approx 1.01\ u.$$

Beta decay increases atomic number by one unit and decreases the number of neutrons by one.

10.4.3 Masses of atoms are measured conveniently in atomic mass units (u). By definition, the Carbon isotope with 6 protons and 6 neutrons and 6 electrons has a mass of 12 amu. Carbon is chosen because it is abundant in natural samples of interest.

$$1u = 1.66 \times 10^{-27}\text{kg} = 931.5\text{Mev}.$$

Thus

$$m_p \approx 1.007276\ u \quad m_n \approx 1.008665\ u, \quad m_e \approx 0.000594\ u.$$

10.5 A mass spectrometer is a device that measures the mass of an atom accurately.

10.5.1 It ionizes the atom, accelerates it through a known potential difference (so that we know its kinetic energy) and then passes it through a known magnetic field. The radius of the orbit depends on the mass; ions of different masses will accumulate at different points on the detector. We can even measures the relative abundance of isotopes in a sample this way.

10.5.2 Another method is just to measure the time of flight after acceleration in a potential. The time taken to go a known distance will give the velocity from which the mass can be determined since you know the kinetic energy.

10.6 Radiocarbon dating (invented by Willard Libby in 1949) is perhaps the most important application of nuclear physics to the rest of science.

10.6.1 The naturally occuring isotopoes of Carbon are C^{12} and C^{13} . In addition the unstable isotope C^{14} with half-life 5730 yrs is also present in trace amounts. This is produced by cosmic rays when they hit the nitrogen in the atmosphere.

$$n + N_7^{14} \to C_6^{14} + p$$

10.6.2 When a plant or animal is still alive, it breaths in air with some fresh C^{14} . Once it dies, it no longer takes in additional C^{14} , so the proportion of this isotope in its remnant will decay exponentially with a half-life of 5730 yrs. By measuring the ratio of C^{14} to the stable isotopes we can work back and determine how long ago the proportions were the same as we see in the atmosphere now. If we postulate that the rate of production of C^{14} remains constant in time, this gives a measure of how long some sample has been dead. The assumption of constant flux of cosmic rays can be tested by comparing with other abundances and has been found to be correct.

10.6.3 Some subtle corrections have been made to this simple picture. The northern and southern hemispheres have slightly different fluxes, because the Earth's magnetic field varies. Also, atmospheric nuclear weapon tests in the fifties changed the concentration of C^{14} .

10.7 Despite these small corrections, the basic technique of radiocarbon dating has been established as a sound pillar of modern science.

10.8 Criticisms of this technique found in popular media are based on obfuscations and misunderstandings of the underlying science.

Chapter 11

Isospin

11.1 The small difference between the neutron and proton masses lead to the idea that they are different states of the same particle, the nucleon, with different values of a new quantum number called isospin.

11.1.1 Since there are only two possible values for this number labeling the neutron and the proton, it was thought to be analogous to the spin of an electron. Isospin means 'like spin' in pidgin greek.

11.2 The nucleon has spin half and isospin half.

11.3 When a neutron and a proton combines into a deuteron, they form an isopin 0 state; because of Fermi statistics, the spin must be 1.

11.3.1 The deuteron has binding energy 2.2 MeV.

11.3.2 The state of a nucleon at rest is a four component complex vector; two such would involve a 4×4 matrix. This matrix must be anti-symmetric because of the exclusion principle, having thus 6 independent states. These can be grouped into sets of three states each that are of spin 1 and isospin zero or isospin one and spin zero. One set of these have an attractive potential and the other must be repulsive. This would explain why there are no nn or pp nuclei.

11.3.3 The α particle is a spin zero and isospin zero state; it can be thought of as a bound state of four nucleons. This has a large binding energy: 28.3 MeV. Whenever there are the right number of neutrons and protons to form an isospin zero state, the binding energy is unusually large: these are called the 'magic nuclei' and they are usually the end products of fission and fusion reactions.

11.4 The electromagnetic interactions do not respect isospin symmetry. In fact, for nucleons,

$$Q = I_3 + \frac{1}{2}$$

where I_3 is isopin.

11.5 The weak interactions also do not respect isospin symmetry. Nuclear beta decay treats the neutron and the proton differently.

11.6 Thus isospin is a symmetry only of strong interactions, which are responsible for the binding of nucleons into nuclei.

11.7 A simple formula of Weiszacker for binding energy is found to be surprisingly accurate:

$$B(N, Z) = a_{vol}A - a_{surface}A^{\frac{2}{3}} - a_{Coul}\frac{Z(Z-1)}{A^{1/3}} - a_{sym}\frac{(N-Z)^2}{A}.$$

for some constants a .

11.7.1 Here, $A = N + Z$ is the mass number; i.e., the total number of nucleons.

11.7.2 The first term is proportional to the number of nucleons; the second to the surface area, as the density of nuclear matter is roughly constant. The third is the Coulomb repulsion and depends on the number of pairs of protons as well as the inverse of the average distance between them. The last term is zero if you have equal numbers of neutrons and protons so that we can form an isospin zero combination.

11.7.3 Iron with $Z = 26, N = 30$ comes close to this.

11.8 The last term can be explained by the postulate that the nuclear force is independent of isospin and spin states of the nucleons. This is related to the $SU(4)$ model of Wigner.

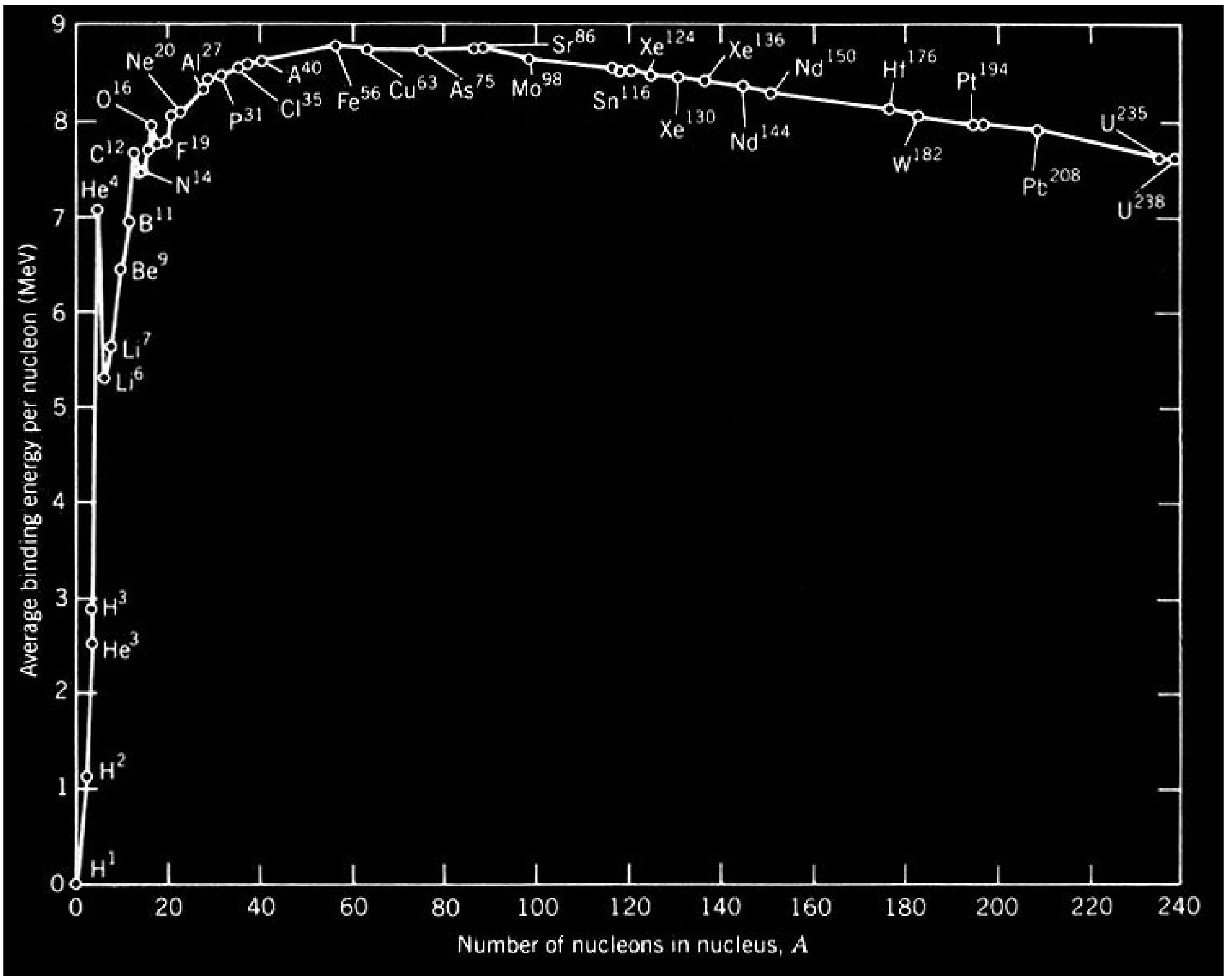

Figure 11.1: Nuclear Binding Energy per Nucleon

Chapter 12

The Deuteron

12.1 The simplest nucleus is the proton. The next simplest is the deuteron, which is a proton bound to a neutron.

12.2 Since the Nucleon is a fermion, the ground state of the two nucleon system must be either $J = 0, I = 1$ or $J = 1, I = 0$.

12.2.1 Here, J is total spin and I the total isospin.

12.2.2 The ground state of most systems have zero orbital angular momentum. This is not quite a general theorem of quantum mechanics, but holds except when the potential is highly singular at zero distance. This almost never happens (has not occured yet) in a real physical system.

12.2.3 Consider first the binding of two spin half particles (ignore isopsin). The ground state wave function is some function of position times a factor involving the spin. The wavefunction must be anti-symmetric by the Pauli principle. If the orbital angular momentum is zero, the positional part is symmetric. (If the orbital angular momentun is l it will change by $(-1)^l$ under an interchange.) Hence the spin part of the wavefunction must be anti-symmetric.

12.2.4 An anti-symmetric combination of spin half wavefunctions has spin zero.

12.2.4.1 To see this, consider two spinors

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

The only antisymmetric combination is

$$\psi\wedge\phi=\frac{1}{\sqrt{2}}[\psi_1\phi_2-\psi_2\phi_1].$$

(The factor of $\frac{1}{\sqrt{2}}$ is to make the length of the product equal to one if ψ,ϕ are normalized to length one.) The angular momentum matrices are direct sums of the individual spin matrices:

$$\Sigma_3[\psi\wedge\phi]=(\sigma_3\psi)\wedge\phi+\psi\wedge(\sigma_3\phi)$$

and similarly for the other components. We can check that they all just vanish. For example

$$\begin{aligned}\Sigma_1[\psi\wedge\phi] &= (\sigma_1\psi)\wedge\phi+\psi\wedge(\sigma_1\phi)\\ &= \frac{1}{\sqrt{2}}[\psi_2\phi_2-\psi_1\phi_1]+\frac{1}{\sqrt{2}}[\psi_1\phi_1-\psi_2\phi_2]\\ &= 0.\end{aligned}$$

12.2.5 A symmetric combination of two spinors can be made in three independent ways:

$$[\psi\vee\phi]_1=\psi_1\phi_1,\quad [\psi\vee\phi]_0=\frac{1}{\sqrt{2}}[\psi_1\phi_2+\psi_2\phi_1],\quad [\psi\vee\phi]_{-1}=\psi_2\phi_2.$$

Again the total spin acts on each component:

$$\Sigma_3[\psi\vee\phi]=(\sigma_3\psi)\vee\phi+\psi\vee(\sigma_3\phi)$$

etc. We can check that

$$\Sigma_3[\psi\vee\phi]_1=[\psi\vee\phi]_1,\quad \Sigma_3[\psi\vee\phi]_0=0,\quad \Sigma_3[\psi\vee\phi]_{-1}=-[\psi\vee\phi]_{-1}$$

so that we get the eigenvalues of spin for a spin $\frac{1}{2}$ particle.

12.2.6 Now consider the nucleon which has both spin and isospin equal to half. To be anti-symmetric in these quantum numbers, the wavefunction can either be (i) symmetric in spin and anti-symmetric in isospin or (ii) anti-symmetric in spin and symmetric in isospin. The first case has spin one and isospin zero and the second has spin zero and isospin one.

12.2.7 Which of these is the ground state depends on the nature of the nuclear force. If the potential energy between nucleons is created by the exchange of a spin zero particle, there would be an attractive force in the channel where the spins of the nucleons are parallel.

12.2.8 The deuteron is observed to have a magnetic moment. This means that it must have non-zero spin; otherwise there would be no preferred direction along which the magnetic moment vector can point. Thus the attractive force is in the spin-one/isopin-zero channel. The force is created by the exchange of a spin zero particle. This particle is called the pion.

12.2.9 We can conclude by the same type of argument as above that the pion must be a spin zero and isospin one particle. It is in effect the orthogonal spin-isospin combination to the deuteron.

12.2.10 Using ideas of relativistic quantum mechanics it can be shown that the potential created by the exchange of a particle of mass m is

$$V(r) = g\frac{e^{-\frac{r}{\lambda}}}{r}, \quad \lambda = \frac{\hbar}{mc}.$$

for some quantity g which is independent of r . This constant has a spin-dependence, ensuring that the spin one combination of nucleons is attractive. This is known as the Yukawa potential, after the scientist who proposed this idea.

12.2.11 Since the range of the interaction is about 1 fm, we can predict that there must a particle which interacts strongly with nuclei with a mass of about 190 MeV. Recall that $\hbar c = 192$ MeV fm.

12.2.12 Also, it must have three isopin states, with electric charges $\pm 1, 0$. This follows from conservation of electric charge in the reactions $n \leftrightarrow p\pi$. Since electromagnetic interactions do not respect isospin, the charged and neutral varieties can have different masses.

12.2.13 Such particles were indeed discovered in cosmic rays. They are called the π mesons. The mass $m_\pi \approx 140$ MeV.

12.2.14 Meson means 'particle in between'. The mesons have masses in between the nucleon and the electron, the only particles that were known at the time of their discovery.

12.2.15 More precisely

$$m_{\pi^\pm} = 139.57018\text{MeV} \ , \quad m_{\pi^0} = 134.9766\text{MeV}$$

12.2.16 The history of there discovery is more complicated. When cosmic ray experiments were looking for the Yukawa particle, they found a particle with about the right mass 105 MeV. It also had electric charges ± 1 . But it didn't interact with the nucleus.

12.2.17 This puzzle was solved by Robert Marshak with his two meson theory. Marshak suggested that the observed particle was not Yukawa's particle, but rather a decay product of it. Now we know that the pions are unstable:

$$\pi^\pm \to \mu^\pm + \text{neutrino}, \quad \text{half} - \text{life} \ = 2.6 \times 10^{-8} \ s$$

and that

$$\pi^0 \to \text{photons}, \quad \text{half} - \text{life} \ = 0.84 \times 10^{-16} \ s$$

The first is a weak decay due to the same forces as the beta decay of the neutron; the second is electromagnetic.

12.3 It is possible to determine the spatial wavefunction of the deuteron by solving the Schrodinger equation for the Yukawa potential numerically.

12.3.1 An analytic solution in terms of standard functions is not possible for the Yukawa potential. It is possible for a very similar potential found by Hulthen . If the range and the strength are properly chosen, there will be exactly one orbital bound state, as observed for the deuteron.

12.3.2 The radial wavefunction of the Schrodinger equation of a pair of particles with a spherically symmetric potential is

$$-\frac{\hbar^2}{2m}\frac{d^2R}{dr^2} + \left[\frac{\hbar^2 l(l+1)}{2mr^2} + V(r)\right] R = ER.$$

12.3.3 Here m is the reduced mass of the two particles. If they have the same mass M then $m = \frac{1}{2}M$. Also r is the distance between the particles.

12.3.4 Thus

$$\frac{d^2R}{dr^2} = \left[-\frac{2mE}{\hbar^2} + \frac{l(l+1)}{r^2} + \frac{2m}{\hbar^2}V(r)\right]R$$

If the potential vanishes at infinite separation,

$$R \approx e^{-kr} \quad \text{as } r \to \infty.$$

12.3.5 The Yukawa potential between nucleons is

$$V_Y(r) = -g\frac{e^{-\mu r}}{r}$$

Here μ is the mass of the pion up to a factor of $\frac{c}{\hbar}$. It is a parameter with dimension of length^{-1}. g is a parameter with the dimensions of energy $\times$ length. The resulting radial equation is

$$\frac{d^2R}{dr^2} = \left[-\frac{2mE}{\hbar^2} + \frac{l(l+1)}{r^2} - \tilde{g}\frac{e^{-\mu r}}{\mu r}\right]R$$

where $\tilde{g} = \frac{2mg\mu}{\hbar^2}$ has dimensions of length^{-2}. This doesn't have an analytic solution.

12.3.6 But we expect the solution for $l = 0$ to behave like e^{-kr} at infinity. The departure from the behavior at infinity is exponential because that is how the potential vanishes. At the origin it must have the same behavior as the radial wave-function of hydrogen. A guess (ansatz) for the wavefunction with these properties is

$$R = \sinh \nu r e^{-kr}.$$

for some constants ν, k .

12.3.6.1 You may think at first that $R(r) \to r^0$ rather than $R(r) \to r$ as $r \to 0$. But if the radial wavefunction tends to a constant, the actual wavfunction $\psi = \frac{1}{r}R$ will diverge at the origin. This means that $\nabla^2\psi \sim \delta(x)$ which is not right. See section 12.6 of Shankar's book.

12.3.7 This can be used a variational ansatz to get the ground state energy of the Yukawa potential. But the Yukawa potential itself is only an approximation. Unlike in atomic physics the nuclear potential is not known exactly: the nucleons are not elementary particles and the fundamental theory of nuclear interactions is at the level of quarks. So we can tinker with the potential as long as it has the correct asymptotic behavior. What is the potential for which the above simple ansatz is the *exact* solution?

12.3.8

$$R'' = \left[k^2 + \nu^2 - 2k\nu \coth \nu r\right] R$$

So comparing,

$$-\frac{2mE}{\hbar^2} + \frac{2m}{\hbar^2}V(r) = k^2 + \nu^2 - 2k\nu \coth \nu r$$

Recall that

$$\coth \nu r = 1 + \frac{2}{e^{2\nu r} - 1}.$$

So

$$-\frac{2mE}{\hbar^2} + \frac{2m}{\hbar^2}V(r) = k^2 + \nu^2 - 2k\nu - \frac{4k\nu}{e^{2\nu r} - 1}$$

Comparing the behavior at the origin

$$\frac{m}{\hbar^2}g = k.$$

Since the potential vanishes at infinity

$$-\frac{2mE}{\hbar^2} = k^2 + \nu^2 - 2k\nu.$$

The rate at which it vanishes must be the same as the Yukawa potential:

$$2\nu = \mu.$$

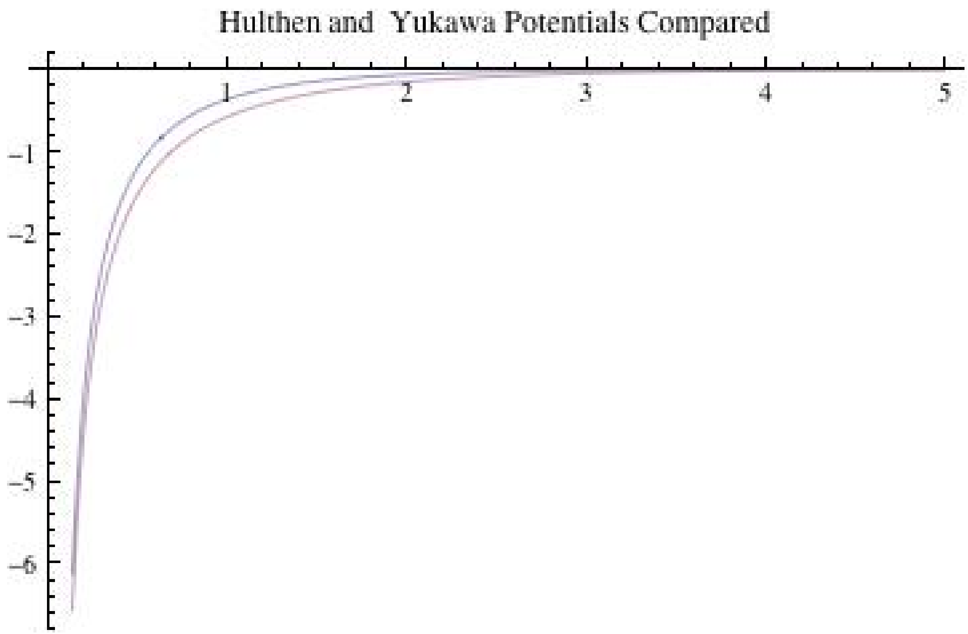

Figure 12.1:

12.3.9 Now we put it all together:

12.4 The ground state solution of the radial equation with the Hulthen potential

$$V_H(r) = -g\mu\frac{1}{e^{\mu r}-1}$$

is

$$R(r) = \sinh\frac{1}{2}\mu r e^{-kr}, \quad k = \frac{m}{\hbar^2}g.$$

with binding energy

$$-E = \frac{\hbar^2}{2m}\left[k - \frac{1}{2}\mu\right]^2$$

12.4.1 This is sufficient to give us an idea of the values of the parameters in the Yukawa potential. We know that

$$-E = 2.2\text{MeV}, \quad mc^2 = 0.5 \times 940\text{MeV}, \quad m_\pi = \mu\hbar c = 140\text{MeV},$$

Then we can solve the quadratic (in natural units $\hbar = c = 1$)

$$-E = \frac{1}{2m}\left[k - \frac{1}{2}\mu\right]^2$$

to get

$$k = \frac{1}{2}\mu + \sqrt{2m|E|}, \quad g = \frac{\mu}{2m} + \sqrt{\frac{2|E|}{m}}.$$

Putting back the needed factors of $\hbar$ and c

$$\frac{g}{\hbar c} \approx .22.$$

12.4.2 This is to be compared to the dimensionless quantity that measures the strength of electromagnetic interactions

$$\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137} \approx 0.0073$$

We can see that the interaction is much stronger. Hence the term **strong interactions** for the nuclear force.

12.4.3 Note that even as $E \to 0$, we have $g \to \frac{\mu}{2g}$; just the fact there is a bound state at all, even with a very small binding energy tells us that there must be a very strong attraction at short range; the strength of the interaction must compensate for its short range.

12.4.4 It turns out that the Yukawa and Hulthen potentials only have a finite number of bound states unlike the Coulomb potential. For the values of the parameters for the deuteron, there is only one bound state. That is in agreement with experiment.

Chapter 13

The Static Quark Model

13.1 A simple picture in which a baryon is made of three quarks explains the charges, masses and magnetic moments to a good approximation.

13.1.1 Let us assume isospin symmetry so that the neutron and proton have the same mass.Each quark would have a mass roughly one-third a nucleon. We are ignoring the binding energy in this simple picture.

13.1.2 If three spin half particles are combined we can get spins $\frac{3}{2}$ or $\frac{1}{2}$. Similarly for isospin. We can see that the proton must be a uud combination while a neutron must be ddu . Thus the charges satisfy

$$2q_u + q_d = 1, \quad 2q_d + q_u = 0, \Rightarrow q_u = \frac{2}{3}, \quad q_d = -\frac{1}{3}.$$

13.1.3 Let us call $\mu_N = \frac{e\hbar}{2m_Nc}$ (where m_N is the mass of a nucleon) the nuclear magneton. This would have been the magnetic moment of the proton if it were an elementary spin half particle. The actual value of the proton magnetic moment is

$$\mu_p \approx 2.8\mu_N.$$

If it were an elementary particle the neutron would have no magnetic moment, since it is neutral. Instead

$$\mu_n \approx -1.8\mu_N.$$

13.1.4 If an up quark is an elementary spin half particle, its magnetic moment is predicted by Dirac's theory to be $\frac{q_u}{m_u}\frac{e\hbar}{2c} = 3q_u\frac{e\hbar}{2m_N c}$. Similarly for the down quark. Thus

$$\mu_u = 2\mu_N, \quad \mu_d = -\mu_N.$$

To combine these into magnetic moments for baryons requires some knowledge of how to add angular momentum in quantum mechanics.

13.1.5 A proton is uud . The two u quarks form a spin combination to which a spin half d quark is added. These can combine into a spin $\frac{3}{2}$ or a spin $\frac{1}{2}$ particle. The first would be a Δ^+ . We can easily predict that the magnetic moment of this state is

$$\mu_{\Delta^+} = 2\mu_u + \mu_d = 3\mu_N.$$

But this is hard to measure.

13.1.6 Think of combining angular momenta j_1 and j_2 . There are $(2j_1 + 1)(2j_2 + 1)$ independent states. They can be labeled by the eigenvalue of the third component of spin for each particle

$$-j_1 \leq m_1 \leq j_2, \quad -j_2 \leq m_2 \leq j_2.$$

But they can also be labelled in terms of the total angular momentum:

$$|j_1 - j_2| \leq j \leq j_1 + j_2, \quad -j \leq m \leq j.$$

There is a unitary transformation that connects these two descriptions.

$$|j, m >= \sum_{m_1, m_2} C^{jm}_{m_1 m_2} |m_1 > |m_2 >$$

These coefficients are called Clebsch-Gordon coeffiecients. It is one of the delights (not!) of quantum theory to calculate these quantities. The most through discussion is in Landau and Lifshitz *Quantum Mechanics*.

13.1.7 The case of interest for us is $j_1 = 1$, $j_2 = \frac{1}{2}$ and $j = \frac{1}{2}, m = \frac{1}{2}$. Clearly the latter state can arise either from $m_1 = 1, m_2 = -\frac{1}{2}$ or $m_1 = 0, m_2 = \frac{1}{2}$. The question is what are the proportion of each combination. Perkins *Intro. High Energy Physics* Appendix C gives the calculations to show that

$$|\frac{1}{2}, \frac{1}{2} >= \sqrt{\frac{2}{3}}|1 > | -\frac{1}{2} > -\sqrt{\frac{1}{3}}|0 > |\frac{1}{2} > .$$

13.1.8 Thus we see that the proton is in the state with $m_1 = 1, m_2 = -\frac{1}{2}$ with probability $\frac{2}{3}$ while in $m_1 = 0, m_2 = \frac{1}{2}$ with probability $\frac{1}{3}$. In the first case its magnetic moment is $2\mu_u - \mu_d$ and in the latter case it is μ_d . Combining these

$$\mu_p = \frac{2}{3}(2\mu_u - \mu_d) + \frac{1}{3}\mu_d = \frac{4}{3}\mu_u - \frac{1}{3}\mu_d = 3\mu_N, \quad \mu_p \approx 2.793\mu_N$$

We compare to the experimental value.

13.1.9 For the neutron we just interchange $u \leftrightarrow d$:

$$\mu_n = \frac{4}{3}\mu_d - \frac{1}{3}\mu_u = -2\mu_N, \quad \mu_p \approx -1.913\mu_N$$

Thus the answer is correct as a first approximation.

13.1.10 In addition to the nucleons, experimentally there also exist the spin $\frac{3}{2}$ particles

$$\Delta^{++}, \Delta^{+}, \Delta^{0}, \Delta^{-}.$$

These fit well as a $I = \frac{3}{2}, J = \frac{3}{2}$ multiplet. Thus for example

$$\Delta^{++} = uuu, \Delta^{+} = uud, \Delta^{0} = udd, \Delta^{-} = ddd.$$

have the right charges. But there is now a puzzle.

13.1.11 How can three identical spin half particles like the u -quarks exist in a state where all of them have parallel spins? (Only way to get spin $\frac{3}{2}$). This seems to violate the exclusion principle. The only possibility is either that they are not fermions, or that there is yet another quantum number we don't know about.

13.1.12 If each quark comes in three 'colors', and only combinations in which the colors are completely antyi-symmetric are allowed, we can explain this puzzle. This turns out to be the correct explanation.

13.1.13 In the decay of $\pi^0 \to 2\gamma$, each kind of quark would contribute a certain amount to the probability amplitude. It turns out that this decay is nine times more likely than we would have expected based on the naive quark model: if each quark exists in three colors, this can be explained as well.

Chapter 14

The Strange Quark

14.1 It was found that K-mesons with mass about 495 MeV were more long-lived than expected i.e., 10^{-8} secs instead f 10^{-23} secs. GellMann and Nishijima proposed that there is a new approximately conserved quantum number to explain this.

14.1.1 Later it was realized that this corresponds to a quark called the strange quark. But due to a historical fluke, the strange quark actually has strangeness -1 . Thus, strangeness is the number of strange anti-quarks minus the number of strange quarks. The K-mesons are

$$K^- = \bar{u}s, \quad K^+ = u\bar{s}, \quad K^0 = d\bar{s}, \quad \bar{K}^0 = s\bar{d}.$$

14.1.2 Thus the s-quark is much like the d-quark; e.g., has electric charge $-\frac{1}{3}$. But it has a greater mass, of about 400 MeV. It is unstable and decays into a u quark by weak interactions, violating strangess. Strong interactions preserve strangeness. This would explain the long lifetimes- the decay is only allowed by the weak interactions.

14.1.3 Weak interactions also allow a transition $K^0 \to \bar{K}^0$. Hence these are not states with definite masses and lifetimes- they are not eigenstates of the hamiltonian. A state that starts out as K^0 can end up and $\bar{K}^0$. The true eigenstates are certain linear combinations called K_S (which has a lifetime of 10^{-10} secs) and K_L (with lifetime 5×10^{-8} secs).

14.2 There are also baryons that carry the strange quark.

14.2.1 They have peculiar names which were given before their quark content was understood.

14.2.2 We will reverse history and start with the spin $\frac{3}{2}$ baryons. Recall that from the u, d quarks we have the four baryons of strangeness zero we will also have some of strangeness-1,-2,-3:

$$\Delta^{++} = uuu \; \Delta^{+} = uud \; \Delta^{0} = udd \; \Delta^{-} = ddd$$

$$\Sigma^{*+} = uus \; \Sigma^{*0} = uds \; \Sigma^{*-} = dds$$

$$\Xi^{*0} = uss \; \Xi^{*-} = dss$$

$$\Omega^{-} = sss.$$

14.2.3 The following are the masses (in MeV) of the 10 spin $\frac{3}{2}$ baryons. The question marks denote values which are theoretically expected, but not measured yet.

$$m_{\Delta^{++}} = 1236 \pm 1, \ m_{\Delta^{+}} = 1234?, \ m_{\Delta^{0}} = 1237 \pm 2, \ m_{\Delta^{-}} = 1244 \pm 8,$$

$$m_{\Sigma^{+*}} = 1382 \pm 1, \ m_{\Sigma^{*0}} = 1380?, \ m_{\Sigma^{*-}} = 1386 \pm 2,$$

$$m_{\Xi^{*0}} = 1529 \pm 2, \ m_{\Xi^{*-}} = 1534 \pm 2$$

$$m_{\Omega^{-}} = 1672 \pm 1.$$

14.2.4 If we assume that the u, d quarks have the same mass (isospin symmetry) but that the strange quark has some mass, we would predict the particles in each line above should have the same mass. This roughly correct. But we can also predict that the average masses on each row must satisfy

$$m_{\Sigma^*} - m_{\Delta} = m_{\Xi^*} - m_{\Sigma^*} = m_{\Omega} - m_{\Xi^*}$$

This is a particular case of the celebrated GellMann-Okubo mass formula.

14.2.5 At that time the masses of the top three rows were known. So the above formula predicts two relations among them which were in fact satisfied. But also it predicts the mass of the last particle, the Ω^- . The observation of this particle in 1961 was spectacular confirmation of the theory of GellMann and Okubo.

14.2.6 Actually, this simple argument using the quark model was not the one used at that time. A more indirect argument using $SU(3)$ symmetry was used, until quarks were accepted as real about ten years later.

14.3 Similar formulae can also be derived for the spin half baryons and the mesons.

14.3.1 For the mesons, the GellMann-Okubo formula involves the squares of the masses, because relativistic wave equation for them (the Klein-Gord equation) has mass^2 instead of mass in it.

14.4 Now we understand that there are altogether six quarks, u, d, c, s, t, b .

Chapter 15

Quarkonium

15.1 The c and b quarks have masses that are large compared to the strong interaction energies but still small compared to the weak interaction energy scale.

15.1.1 Strong interactions involve energies of a few hundred MeV (the mass of the pion, the binding energy of mesons etc.) while weak interactions are small at energies less than the mass of the W, Z which are ab out 100 GeV. The mass of the charm quark is about 1.5 GeV and that of the b quark about 5 GeV. The u, d, s quarks have masses small enough that their strong interaction binding energies are as big as their masses. The t quark with a mass of 180 GeV decays to W, Z before it has time to form bound states by the strong interactions. Thus the strong binding of c, b is a special case where Non-Relatvistic Quantum Mechanics can be applied. A rare opportunity not to be missed.

15.2 A good approximation to the masses of the $c\bar{c}$ and $b\bar{b}$ mesons can be obtained by solving the Schrodinger equation for the potential

$$V(r) = kr - \frac{a}{r}$$

15.2.1 The Coulomb force is due to gluons; that due to photons can be ignored as it is ten times smaller. $a \sim 0.1$, $\alpha \sim 10^{-2}$.

15.3 The Schrodinger equation for the linear potential can be solved in terms of the Airy function. The masses of the bound states with $l = 0$ are given by the formula

$$M_n = 2m + \left[\frac{b^2}{m}\right]^{\frac{1}{3}} a_n$$

where the $-a_n$ are simple roots of the Airy function.

$$\text{Ai}(-a_n) = 0, \quad a_n = 2.338, 4.088, 5.521, 6.787, 7.944 \cdots$$

15.3.1 This is in units with $\hbar = c = 1$. So V has dimensions of mass, a is dimensionless and b has dimensions of m^2 . That explains the cube-root.

15.3.2 Also, m is the mass of the quark.

15.3.3 From the masses $M_\psi = 3.105$ GeV and $M'_\psi = 3.695$ GeV, we get the mass of the charm quark and the string tension

$$m = 1.16 \text{ GeV}, b = 0.211 \text{ GeV}^2.$$

From this the masses of the remaining excitations of zero orbital angular momentum can be obtained.

15.3.3.1 The Airy function is the unique (up to multiplicative constant) solution of the differential solution

$$y'' - xy = 0$$

which vanishes as $x \to \infty$. It is possible to get a formula for it as a power series in x and as a Fourier transform. It has simple zeroes along the negative real axis.

15.3.3.2 The radial Schrodinger equation for a spherically symmetric potential equation

$$\frac{1}{m} R'' + \left[E - \frac{l(l+1)}{r^2} - V(r)\right] R = 0, \quad \psi(r, \theta, \phi) = \frac{1}{r} R Y_{lm}(\theta, \phi).$$

15.4 Similar results can be found for the Υ states which are $b\bar{b}$ bound states.

15.4.1 Inclusion of orbital angular momentum, spin-spin coupling, Coulomb interaction due to gluons can also be done and give a satisfactory picture of quarkonium spectroscopy: not only masses but also the transitions among the states can be understood.

Chapter 16

Symmetry Breaking

16.1 If a particle moves under the influence of a potential, its energy is given by

$$H = \frac{p^2}{2m} + V(q)$$

where p, q are momentum and position vectors.

16.1.1 The state of lowest energy is $p = 0$ and q_0, the point at which V is a minimum:

$$\frac{\partial V}{\partial q} = 0.$$

Let us assume for the moment that the minimum is isolated. That is, there is no nearby point to q_0 which is also a minimum. In this case the second derivative

$$K_{ij} = \left[\frac{\partial^2 V}{\partial q_i \partial q_j}\right]_{q=q_0}$$

is a positive matrix:

$$u^T K u > 0, \quad \text{for all } u.$$

16.1.2 If the particle is pulled away from the minimum, it is pulled back and it will oscillate around q_0. The frequency of these oscillations are given by the eigenvalues k_i of K.

$$\omega_i^2 = \sqrt{\frac{k_i}{m}}.$$

16.1.3 Quantum mechanics says that to get away from the ground state, it will cost a minimum of $\hbar\omega_{\text{min}}$ of energy. If you don't have that much energy, you are stuck at the ground state. There is a 'gap' in the energy spectrum.

16.2 If the minimum is not isolated, K can have zero eigenvalues.For small energies, the particles will move around among the minima instead of oscillating.

16.2.1 Quantum mechanically, there are states with energy as close to the ground state as possible. There is no 'gap' in the spectrum.

16.3 A particle in a potential $V(q)$

$$V(q) = \lambda(q^2 - a^2), \quad \lambda > 0.$$

has a minimum on a sphere of radius a

16.3.1 All the directions of q have the same energy.

16.3.2 If the particle is moving in this potential with some loss of energy (due to friction or radiation) it will settle down eventually at one of the points on the sphere. A slow movement of the particle will cost very little energy, as long as it stays on the sphere. We can make the energy as small as we want by making the particle move slow enough.

16.3.3 To leave the sphere, it will cost some minimum amount of energy, no matter how slow you move.

16.3.4 We experience something like this on the surface of the Earth, due to gravity. To move horizontally, it costs very little energy. To move vertically, is harder.

16.3.5 This situation where there are many states of minimum energy causes the phenomenon of symmetry breaking. Whenever the system chooses a direction among many equivalent ones, a zero-frequency mode becomes possible.

16.3.6 For the above potential, where q is a vector with two components, the matrix K will be of the form

$$K = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}$$

for some positive constant k. Thus there is an eigenvector of zero eigenvalue and one with non-zero eigenvalue.

16.4 If we break the rotational symmetry by adding a small term that depends on the direction, K will not have zero eigenvalues any more; but two of the eigenvalues will be much smaller than that the other one.

16.4.1 Consider

$$V(q) = \lambda(q^2 - a^2)^2 + \epsilon q_1$$

for a small positive ϵ. The points on the circle don't all have the same energy.

16.4.2 Exercise: For the above potential, the position of the minimum for small ϵ. Expand around this position and determine the frequencies to first order in ϵ.

16.4.3 This kind of situation happens with magnets.A magnet is made up of a large number of molecules, each with small magnetic moment. It costs less energy if they all point in the same direction. Once a direction is chosen, change in the direction will cost very little energy, as log as it is slow enough. If the magnetization at each point is given by a vector M, the energy of a small change is

$$E = \int [(\nabla M)^2 + \lambda(M^2 - a^2)^2] dx$$

16.4.4 If we can move the direction of all the magnets in unison, it will cost very little energy. The movement of magnetization moves through the system with a constant speed. This speed is a property of the material, similar to the speed of sound. Such 'magnetic waves' and the corresponding particles called 'magnons' have been observed, confirming this theory.

16.5 This is the Landau-Ginzberg theory of magnetization.

16.6 Nambu had the idea that a similar theory might explain why the pi mesons are almost massless.

16.6.1 There could be some symmetry whose breaking would cause there to be three waves (massless) moving at the velocity of light. The speed is not any more determined by properties of a material, but be a constant of nature: the speed of light.

Chapter 17

Variational Principles

17.1 All the basic laws of physics can be expressed as Partial Differential Equations; these can be thought of the condition for a quantity called the action to be an extremum.

17.1.1 For example, the harmonic oscillator

$$\ddot{q} + \omega^2 q = 0.$$

follows from minimizing the action

$$S[q] = \int_{t_1}^{t_2} [\frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2] dt$$

Note that S depends on the whole function $q(t)$ for $t_1 < t < t_2$; not just at one particular time. Thus it depends on an infinite number of variables. If we require that S be an extremum under all changes to q that keep the boundary values at t_1, t_2 fixed, we will get the harmonic oscillator equation.

$$\delta S = \int_{t_1}^{t_2} [\dot{q}\frac{d}{dt}\delta q - \omega^2 q \delta q] dt$$

Integrating by parts and using the fact that δq vanishes at the boundary, we get

$$\delta S = \int_{t_1}^{t_2} [-\ddot{q}\delta q - \omega^2 q]\delta q dt$$

This vanishes precisely when the equations are satisfied.

17.1.2 More generally,

$$S[q] = \int_{t_1}^{t_2} [\frac{1}{2}\dot{q}^2 - V(q)]dt$$

leads to

$$\ddot{q} = -\frac{\partial V}{\partial q}.$$

17.2 The Klein-Gordon equation for a massive particle is a generalization of the harmonic oscillator, where time is replaced by space-time:

$$\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi + m^2 \phi = 0.$$

It follows from the variational principle

$$S = \frac{1}{2}\int [\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2] d^3x dt$$

17.2.1 Again we vary ϕ keeping its initial and final values fixed. Except that now these are specified at surfaces $t = t_1$ and $t = t_2$ in space-time.

17.2.2 We choose units with $c = 1$.

17.3 Nonlinear wave equations follow if we choose instead an action with another potential $V(\phi)$:

$$S = \frac{1}{2}\int [\dot{\phi}^2 - (\nabla\phi)^2 - V(\phi)] d^3x dt$$

$$\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi + \frac{\partial V}{\partial \phi} = 0.$$

17.4 If we have a particle with many internal states, the field can have many components

$$S = \frac{1}{2}\int [\sum_a (\dot{\phi}_a^2 - (\nabla\phi_a)^2) - V(\phi)] d^3x dt$$

$$\frac{\partial^2\phi_a}{\partial t^2} - \nabla^2\phi_a + \frac{\partial V}{\partial \phi_a} = 0.$$

17.5 If the potential has a minimum at some point v, we expand around that minimum $\phi_a = v_a + \chi_a$ to get a collection of massive particles:

$$\frac{\partial^2\chi_a}{\partial t^2} - \nabla^2\chi_a + \sum_b K_{ab}\chi_b + ... = 0, \quad K_{ab} = \left[\frac{\partial^2 V}{\partial\phi_a \phi_b}\right]_{\phi=v}$$

The square of the masses of the particle are the eigenvalues of the symmetric matrix K.

17.5.1 These will interact with each other due to the higher order terms we ignored.

17.5.2 The equation may have other solutions as well, that are not just small oscillations around the minimum. These will appear to be additional particles predicted by the same action principle.

17.6 If the minimum is not isolated, the matrix K may have zero eigenvalues. Then there will be massless particles in the theory.

17.6.1 This happens when the potential has some symmetry, which is broken by the choice of v that we are expanding around. In the linear sigma model, the field has four components. The fourth component used to be called σ, so this is called the "sigma model".

$$V(\phi) = \lambda[\phi_1^2 + \phi_2^2 + \phi_3^3 + \phi_4^2 - v^2]^2.$$

The minimum of V is not a point: it is a sphere of radius v in four dimensional space. The matrix K has three eigenvectors with eigenvalue zero; and one with positive eigenvalue. These three massless particles can be identified with the π mesons.

17.6.2 If we add a small term that lifts the degeneracy of the minima, so that the minimum is isolated, we get three particles of small masses plus one of large mass:

$$S = \frac{1}{2}\int[\sum_a(\dot{\phi}_a^2 - (\nabla\phi_a)^2) - V(\phi), \quad V(\phi) = \frac{\lambda}{2}[\phi_1^2 + \phi_2^2 + \phi_3^3 + \phi_4^2 - v^2]^2 + \epsilon\phi_4.$$

Exercise Find the minimum of this potential. Find the matrix K of second derivatives and its eigenvalues.

17.6.3 Choosing $\epsilon > 0$ but small, we will get three massive particles of equal mass and one particle of much large mass.

17.6.4 The rotation symmetry among the four directions is broken in two ways here. When $\epsilon = 0$, the energy is the same in all directions on the sphere $|\phi| = v$; but we choose one of them to expand around, leading to zero mass particles. This is called spontaneous symmetry breaking. Whenever a symmetry is broken this way there are massless particles, called Nambu-Goldstone bosons.

17.6.5 When ϵ is not zero but small, we give a special status to the fourth direction. So the symmetry is broken from the four dimensional rotation group to the three dimensional one. This is called explicit breaking of the symmetry. The Nambu-Goldstone bosons aquire a small mass, equal to $\sqrt{\frac{\epsilon}{v}}$. The idea was original developed to account for the small mass of the pi mesons.

17.6.6 The particle σ is so heavy that sometimes it is useful to consider the limit where it has infinite mass. In this limit the field ϕ takes values on the sphere $|\phi| = v$. The action becomes

$$S = \frac{1}{2}v^2\int g_{ab}(\theta)\partial_\mu\theta^a\partial^\mu\theta^b d^4x$$

where θ^a are curvilinear co-ordinates on the sphere and g_{ab} is a Riemann metric on it. The equations of motion of that follow are nonlinear and is called the "wave map equation".

$$\partial^\mu[g_{ab}\partial_\mu\theta^b] = \frac{\partial g_{bc}}{\partial\theta^a}\partial_\mu\theta^b\partial^\mu\theta^c$$

If θ is small this is the wave equation; the nonlinearities describe interactions among the pi mesons.