PHY402 Probability Spring 2006

Problem Set 1 Due Wednesday Feb 1

1 Find the mean and standard deviation of the Poisson distribution

$$P(r) = \frac{\lambda^r}{r!} e^{-\lambda}, \quad r = 0, 1, \cdots, \quad \lambda > 0. \tag{1}$$

Then extend the calculation to find $< r^3 >$.

2 We saw that the limit of the binomial distrubition $P_N(r) = \begin{pmatrix} N \\ r \end{pmatrix} p^r q^{N-r}$ as $N \to \infty$ keeping $\lambda = Np$ fixed is the Poisson distribution $P(r) = \frac{\lambda^r}{r!} e^{-\lambda}$. What is the size of the error in this approximation? (i.e., estimate the terms we ignored in making the approximation).

3 Use a calculator or short computer program to calculate the factorials of numbers $1, 2, \cdots 20$. Calculate also the Stirling approximation to the factorial. How large should n be in order that the error in the Stirling approximation for $n!$ be no larger than 10%?; also 1%? What is the error in the worst possible case $n = 1$?

PHY402 Probability Spring 2006

Problem Set 2 Due Wednesday Feb 8

4 Use appropriate limiting cases of the binomial distribution in the following problems.

1. It is known that the O-negative blood type is found in 7% of people. What is the probability that in a random sample of 30 people, the number with this type is more than 3? What is the probability there will be at least one person with this blood type in a random sample of 15 people?

2. The proportions of males and females in the human population are equal. What is the probability that in a random sample of a 100 people, more than 55 will be male? What is the probability that there will be only 4 women in a group of 15 in a random sample?

5 Suppose a particle executes a biased random walk on a line: at each step it moves either to the right or to the left a distance l with probabilities p and $q = 1 - p$ respectively. (These probabilities may *not* be equal). The number of such steps in unit time is n.

1. What is the average velocity; i.e., the rate at which the average position changes with time?

2. What is the variance of the position as a function of time?

3. Find the limiting probability distribution for position at time t assuming that l is small but that n is large; i.e., a large number of small steps.

6 If a random variable x is changed to $x \mapsto ax+b$ (where a and b are constants), how do the mean and variance change? What is the function of x that has zero mean and unit variance?

PHY402 Probability Spring 2006

Problem Set 3 Due Wednesday Feb 15

7 **7.1** A Geiger counter emits a 'click' each time a radioactive decay happens. If the average number of decays in unit time is λ, what is the probability distribution of the time interval between two successive clicks?

Hint What is the probability that there are no decays in the interval $[0, t]$ *and* there is a decay in the interval $[t, t + dt]$?

7.2 Assume that homes in the prairie are distributed uniformly with an average density of n per square mile. What is the probability distribution of the distance to the nearest neighbor from a given home?What is the average distance between nearest neighbors?

8 Most computer languages have a built in function that will produce a random variable distributed uniformly in the range $[0, 1]$. How will you use it to generate random variables that are distributed according to the (i) exponential law $p(y) = e^{-y}$ (ii) The Gaussian law $p(y) = \frac{e^{-\frac{y^2}{2}}}{\sqrt(2\pi)}$ or (iii) the Cauchy or Lorentzian law $p(y) = \frac{1}{\pi}\frac{1}{1+y^2}$.? Verify your answers with a numerical simulation; i.e., generate about a thousand random numbers and plot the relative frequencies to compare with the predictions.

9 Consider the following simple model for the size of a colony of bacteria. We start with a number n_0; in each generation the number can be either be multiplied by a factor u with probability 0.5 or divided by u with probability 0.5. (Here $u-1$ is a small positive number).What is the probability distribution of the number of bacteria after a large number N of steps?

PHY402 Probability Spring 2006

Problem Set 4 Due Wednesday Feb 22

10 **10.1** If $\eta = a\xi + b$ where a and b are constants, what is the relation between the characteristic function of ξ and that of η?

10.2 Find the characteristic function of the Gaussian distribution

$$p_\xi(x) = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{(2\pi)}\sigma} \tag{2}$$

Find the first four moments explicitly as functions of μ and σ.

11 Find the characteristic function of the Cauchy (also called Lorentz) distribution:

$$p_\xi(x) = \frac{1}{\pi}\frac{1}{1+x^2}. \tag{3}$$

Show that all the odd moments are zero and that all the even moments (except μ_0) are infinite. What does this mean in terms of the characteristic function?

12 Assume that the angle ϕ can take any value between 0 and 2π with equal probability. What are the probability distributions of $x = \cos\phi$ and $y = \sin\phi$? What is the joint distribution of x and y ? Are they statisticaly independent?

PHY402 Probability Spring 2006

Problem Set 5 Due Wednesday Mar 1

13 $\xi_1, \cdots \xi_n$ are a set of n independent identically distributed random variables uniformly distributed in the interval $[0, 1]$. Let μ and σ be the mean and standard deviation of one of them. Define $\zeta_n = \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{n} [\xi_i - \mu]$

13.1 What is the probability density function of ζ_n for $n = 1, 2, 3$?

13.2 What is the probability density function of ζ_n as $n \to \infty$?

14 Write a computer program that generates n uniform random variables, and calculates ζ_n as defined above. Then repeat this a large number N times to get a simutaled data sample for ζ_n and compute the relative frequencies in bins of equal width.(Your use judgement to choose the width and the number of bins.) For $N = 1000$ and $n = 1, 2, 3, 100, 1000$ compare the simulation with the theoretical predictions above, by plotting or by preparing a table.

15 Let ξ_1, ξ_2 be independent identically distributed random variables with the Cauchy (or Lorentzian) probability density function:

$$p_{\xi_{1,2}}(x) = \frac{1}{\pi} \frac{1}{1 + x^2} \tag{4}$$

What is the probability density function of the sum $\xi_1 + \xi_2$? If there are n such independent identically distributed Cauchy variables $\xi_1, \cdots \xi_n$ what is the probability density function of their average

$$\zeta_n = \frac{1}{n}[\xi_1 + \cdots \xi_n] \tag{5}$$

in the limit as $n \to \infty$?

PHY402 Probability Spring 2006

Problem Set 6 Due Wednesday Mar 8

16 Let ξ_1, ξ_2 be two independent Gaussian random variables, both of zero mean and unit variance. What are the probability densities of the variables

$$\rho = \frac{1}{2}(\xi_1^2 + \xi_2^2), \quad \theta = \arctan\frac{\xi_2}{\xi_1}? \tag{6}$$

Hint Think geometrically; what are the countour curves on which the probability density is constant?

17 Let η_1, η_2 be independent random variables with uniform (constant) probability density in the range $[0, 1]$. Find changes of variables $\xi_1 = f_1(\eta_1, \eta_2), \xi_2 = f_2(\eta_1, \eta_2)$ which produces a pair of independent Gaussian random variables of zero mean and unit variance.

Hint Use the answer to the previous problem. You should be able to do this using elementary functions such as the exponential, trigonometric and algebraic functions; i.e., not the error function.

18 Generate a data set of a thousand values of a Gaussian random variable of zero mean and unit standard deviation, starting from the built in function in most computer languages that produces uniform variables in the range $[0, 1]$. Plot a histogram of the relative frequencies of this data set against the predicted probability density function.

PHY402 Probability Spring 2006

Problem Set 7 Due Friday Mar 17

Have a Good Spring Break!

PHY404 Linear Spaces Spring 2006

Problem Set 8 Due Wed Apr 12

19 Let $A = \begin{pmatrix} a & b \\ b^* & d \end{pmatrix}$ where a, d are real numbers and b is some complex number.

19.1 Find the eigenvalues and corresponding eigenvectors of A

19.2 What are the conditions on a, d, b in order that the two eigenvalues coincide?

20 Let A be as in the last problem.

20.1 Find the resolvent of the A.

20.2 Find the positions of the singularities of this resolvent and the residues at these singularities.

20.3 What is the precise relationshiop between them and the eigenvalues and eigenvectors found in the last problem?

21 Let V be the space of all complex-valued functions of a real variable, with period 2π.

21.1 Find a basis for such functions. What is the dimension of this space?

21.2 For what values of λ are there solutions in V to the eigenvalue equation

$$\frac{d^2}{dx^2}\psi = \lambda\psi? \tag{7}$$

21.3 Find a basis of solutions for each such value of λ.

PHY404 Linear Spaces Spring 2006

Problem Set 9 Due Wed Apr 19

22 Let $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Find the minimum of the function $f(u) =< u|\sigma_1 u >$ over all *real* vectors of length one. Verify that the minimum value is an eigenvalue of σ_1 and that the position of the minimum is an eigenvector. Similarly find the maximum and interpret it in terms of an eigenvector and eigenvalue.

23 Let A be a finite dimensional linear operator (not necessarily hermitean). Suppose that we have eigenvectors for $A, A^\dagger$:

$$A\psi = \lambda\psi \quad A^\dagger\chi = \mu\chi. \tag{8}$$

Show that $< \chi|\psi >= 0$ if $\lambda \neq \mu^*$.

23.1 For hermitean operators (i.e., $A = A^\dagger$) show that all the eigenvalues are real and that the eigenvectors with unequal eigenvalues are orthogonal.

24 Define the shift operators

$$A^\dagger|n >= |n{+}1 > \text{ for } n = 0, 1, \cdots, \quad A|n >= |n{-}1 > \text{ for } n = 1, 2\cdots, \quad A|0 >= 0. \tag{9}$$

where the collection of vectors $|n >, n = 0, 1, \cdots$ is an orthonormal basis.

24.1 For any complex number z with $|z| < 1$, find an eigenvector for A with eigenvalue z, as a linear combination $\sum_{n=0}^{\infty} c_n(z)|n >$. Find the length of the eigenvector by evaluating the sum $\sum_0^\infty |c_n(z)|^2$ and use that to find an eigenvector of unit length.

24.2 Does $A^\dagger$ have any eigenvectors of finite length?

PHY404 Linear Spaces Spring 2006

Problem Set 10 Due Wed Apr 26

25 Find the resolvents of the matrices

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{10}$$

25.1 Find the position and order of their singularities.

25.2 Evaluate the integrals $\int z^n R(z)dz$ around a contour containing each singularity, for $n = 0, 1, \cdots m-1$ where m is the order of the singularity and $R(z)$ is the resolvent.

25.3 Find the canonical decomposition of each matrix (i.e., write it as a sum of projections to eigenspaces and generalized eigenspaces.)

26 Let A be a hermitean matrix and $R(z) = (A-z)^{-1}$ its resolvent. Find a bound on $|R(z)|$ and use that to show that the eigenvalues of A are real.

27 Again, let A be a hermitean matrix and $R(z)$ its resolvent.

27.1 What is the significance of the quantity$-\frac{1}{2\pi i}\int_C R(z)dz$ where C is some contour in the complex plane that does not pass through any singularities of $R(z)$?

27.2 Show that

$$A^n = -\frac{1}{2\pi i}\int_D z^n R(z)dz \tag{11}$$

where D is a contour that contains inside it all the singularities of $R(z)$.