

**PHY405 Geometrical Methods in Physics Spring 2004 S. G. Rajeev**

**Problem Set 1**

**1** Let  $V$  be a real vector space and  $V'$  its dual (space of linear functions valued in  $R$ ). Show that the dual of the dual,  $(V')'$  is contained in  $V$ . If  $V$  is finite dimensional, find a basis in  $V'$  given a basis  $e_1 \cdots e_n$  in  $V$ .

**2** A positive inner product is a bilinear function  $g : V \times V \rightarrow R$  which is symmetric and positive:

$$g(u, u) \geq 0 \quad \text{and} \quad g(u, u) = 0 \iff u = 0. \quad (1)$$

Prove the Schwartz inequality

$$|g(u, v)|^2 \leq g(u, u)g(v, v) \quad (2)$$

for any pair of vectors in the real vector space  $V$ .

**3** Find the formula for the metric tensor and the Laplace operator in  $R^2$  in polar co-ordinates

$$x = r \cos \theta, \quad y = r \sin \theta. \quad (3)$$

**3.1** Show that the Laplace equation is separable in these co-ordinates.

**3.2** What is the most general potential  $V(r, \theta)$  for which the Schrodinger equation is separable in these co-ordinates?

**4** Show that the Schrodinger equation for the hydrogen molecular ion  $H_2^+$  is separable in elliptic co-ordinates.

$$x = a \cosh \mu \cos \theta, \quad y = a \sinh \mu \sin \theta. \quad (4)$$

There are two protons at rest with a single electron orbiting them. Assume for simplicity that the wave function is invariant under rotations around the axis connecting the two protons. You have to find the potential energy and kinetic energy of the electron in elliptic co-ordinates; show that the wave equation can be reduced to ordinary differential equations. **This is the hardest of this set of problems. Do the best you can but don't get too worried if you can't do it all yet.**

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**Problem Set 2 Due 25 Feb 2008**

**5** The metric on the sphere is, in polar co-ordinates,  $ds^2 = d\theta^2 + \sin^2 \theta d\phi^2$ .

**5.1** Show, by finding an appropriate change of co-ordinates, that it can be expressed as  $ds^2 = 4 \frac{dzd\bar{z}}{(1+z\bar{z})^2}$ .

**5.2** Show that the transformation

$$M(z) = \frac{az + b}{-\bar{b}z + \bar{a}}, \quad |a|^2 + |b|^2 = 1 \quad (5)$$

is a symmetry of the above metric.

**5.3** What is the relation of this transformation to a rotation?

**6** The metric on hyperbolic space is  $ds^2 = d\theta^2 + \sinh^2 \theta d\phi^2$ .

**6.1** Find the transformations that takes metric to the form

$$ds^2 = 4 \frac{dw_1^2 + dw_2^2}{w_2^2}, \quad w_2 > 0 \quad (6)$$

**6.2** Show that the *Mobius transformation* defined by real numbers  $a, b, c, d$

$$M(w) = \frac{aw + b}{cw + d}, \quad w = w_1 + iw_2, \quad ad - bc = 1 \quad (7)$$

is a symmetry of the above metric.

**6.3** Show that any point on the upper half plane can be mapped to  $i$  using a Mobius transformation;

**6.4** And that there is a transformation which maps any pair of points  $P_1, P_2$  to  $i$  and  $iy$  for some real number  $y > 0$ .

**7** Derive the geodesic equation for the hyperbolic metric (6) on the upper half plane.

**7.1** For two points that lie on the imaginary axis, show that the geodesic of the hyperbolic metric is just the straight-line between them. What is the distance of a point to the origin? (Contd. next page)

**7.2** By using the Möbius transformation above, find the geodesic between a pair of arbitrary points.

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**Problem Set 3 Due 31 Mar 2008**

**8** Derive the conditions for a linear transformation  $\tilde{x}^\mu = \Lambda^\mu_\nu x^\nu$  to preserve the Minkowski metric. These are called Lorentz transformations.

**8.1** Show that the Lorentz transformation that involves only  $x^0$  and  $x^1$  is of the form

$$\Lambda(\theta) = \begin{pmatrix} \cosh \theta & \sinh \theta & 0 & 0 \\ \sinh \theta & \cosh \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (8)$$

What is the relation of  $\theta$  to velocity?

**8.2** If we compose two such transformations  $\Lambda(\theta_1)$  and  $\Lambda(\theta_2)$ , what is the result? Derive the relativistic law of addition of velocities from this.

**9** The momentum and energy of a particle combine to form a four-vector  $p_\mu$  in space-time.

**9.1** Show that the relativistic relation between energy and momentum is (in units with  $c = 1$ )

$$\eta^{\mu\nu} p_\mu p_\nu = m^2 \quad (9)$$

In the limit of small kinetic energy ( compared to  $m$ ) how does this reduce to usual formula in Newtonian physics?

**9.2** Using the usual rules of quantum mechanics

$$\mathbf{p} = -i\hbar\nabla, \quad E = i\hbar\frac{\partial}{\partial t} \quad (10)$$

derive the relativistic analogue of the Schrodinger equation for a free particle of mass  $m$ :

$$\hbar^2 \eta^{\mu\nu} \partial_\mu \partial_\nu \psi + m^2 \psi = 0. \quad (11)$$

This is called the Klein-Gordon equation.

**10** Define a 2-form  $F = F_{\mu\nu} \frac{dx^\mu \wedge dx^\nu}{2}$  on space-time with  $F_{0i} = E_i$  the electric field and  $F_{12} = B_3, F_{23} = B_1, F_{31} = B_2$  being the magnetic field.

**10.1** Show that the source less Maxwell's equations can be written as  $dF = 0$  or equivalently as

$$\partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} + \partial_\sigma F_{\mu\nu} = 0. \quad (12)$$

**10.2** Show that the above equations are solved by  $F = dA$  or

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (13)$$

for some one-form  $A$ . What is the physical meaning of the components of  $A$ ?

**10.3** .

In terms of the Minkowski metric, show that the remaining Maxwell's equations become

$$\eta^{\mu\nu} \partial_\mu F_{\nu\sigma} = j_\sigma \quad (14)$$

What do the components of  $j$  mean physically?

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Problem Set 4 Due 14 April 2008

11  $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \cdot \\ \cdot \\ \psi_n \end{pmatrix}$  is a scalar field on space-time with  $n$  complex components.

11.1 Under a gauge transformation  $\psi(x) \mapsto g(x)\psi(x)$ , where  $g(x)$  is a unitary matrix, show that the derivative transforms as  $\psi \mapsto g[\partial_\mu\psi + g^{-1}\partial_\mu g\psi]$

11.2 Let  $A_\mu(x)$  be a vector field whose components are anti-hermitean matrices. How should  $A_\mu$  transform in order that the covariant derivative

$$\nabla_\mu\psi = \partial_\mu\psi + A_\mu\psi \quad (15)$$

transform as  $\nabla_\mu\psi \mapsto g[\nabla_\mu\psi]$  under a gauge transformation?

11.3 Show that  $\nabla_\mu\nabla_\nu\psi - \nabla_\nu\nabla_\mu\psi$  does not involve any derivatives of  $\psi$ . Define the *curvature*  $F_{\mu\nu}$  by

$$\nabla_\mu\nabla_\nu\psi - \nabla_\nu\nabla_\mu\psi = F_{\mu\nu}\psi. \quad (16)$$

Obtain a formula for  $F_{\mu\nu}$  in terms of  $A_\mu$ .

12 How does  $F_{\mu\nu}$  transform under gauge transformations?

12.1 Find a definition of covariant derivative for  $F_{\mu\nu}$  such that under a gauge transformation

$$\nabla_\rho F_{\mu\nu} \mapsto g[\nabla_\rho F_{\mu\nu}]g^{-1} \quad (17)$$

12.2 Show the identity

$$\nabla_\rho F_{\mu\nu} + \nabla_\mu F_{\nu\rho} + \nabla_\nu F_{\rho\mu} = 0. \quad (18)$$

12.3 Show that the Yang-Mills field equations are gauge invariant:

$$\nabla^\mu F_{\mu\nu} = 0. \quad (19)$$

**13** Show that in the special case  $n = 1$  show that the above equations (18,19) reduce to Maxwell's equations.

**13.1** In the general case, show that to leading order in  $A_\mu$ , the Yang-Mills theory describes a collection of  $n^2$  massless particles, analogous to the photon; i.e., that if we ignore the non-linear terms, the  $n^2$  components of  $A_\mu$  satisfy Maxwell's equations.

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**Problem Set 5 Due 30 April 2008**

**14** Define the covariant derivative by

$$\nabla\phi = \partial_\mu\phi, \quad \nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma_{\mu\rho}^\nu v^\rho, \quad \nabla_\mu v_\nu = \partial_\mu v_\nu - \Gamma_{\mu\nu}^\rho v_\rho. \quad (20)$$

Extend it to tensor fields in the natural way, thinking of them as linear combination of products of vector fields. Postulate zero torsion and the preservation of the metric:

$$\nabla_\mu \nabla_\nu \phi = \nabla_\nu \nabla_\mu \phi, \quad \nabla_\mu g_{\nu\rho} = 0. \quad (21)$$

**14.1** Derive a formula for the Christoffel symbols  $\Gamma_{\mu\nu}^\rho$  in terms of the partial derivatives of the metric tensor.

**14.2** Show that the anti-symmetric part of the second covariant derivative does not involve derivatives of the vector field. Define the curvature tensor by

$$\nabla_\sigma \nabla_\alpha v^\mu - \nabla_\alpha \nabla_\sigma v^\mu = R_{\sigma\alpha\beta}^\mu v^\beta \quad (22)$$

Derive a formula for  $R_{\sigma\alpha\beta}^\mu$  in terms of derivatives of  $\Gamma_{\beta\gamma}^\alpha$ . Note the analogy with the definition of the Yang-Mills field strength.

**15** Derive the following identities for the curvature tensor. Some are obvious, others require a lot of calculation to verify

$$\begin{aligned} R_{\sigma\alpha\beta}^\mu &= -R_{\alpha\sigma\beta}^\mu \\ R_{\sigma\alpha\beta\mu} &= R_{\beta\mu\sigma\alpha}, \text{ where } R_{\sigma\alpha\beta\mu} = R_{\sigma\alpha\beta}^\nu g_{\mu\nu} \\ R_{\sigma\alpha\beta\mu} &= -R_{\sigma\alpha\mu\beta} \\ R_{\sigma\alpha\beta}^\mu + R_{\alpha\beta\sigma}^\mu + R_{\beta\sigma\alpha}^\mu &= 0 \\ \nabla_\nu R_{\sigma\alpha\beta}^\mu + \nabla_\alpha R_{\nu\sigma\beta}^\mu + \nabla_\sigma R_{\alpha\nu\beta}^\mu &= 0 \end{aligned} \quad (23)$$

The last one is called the Bianchi identity. What is its analogue for Yang-Mills theory?

**16** Define the Ricci tensor and the Ricci scalar by

$$R_{\sigma\beta} = R_{\sigma\alpha\beta}^{\alpha}, \quad R = g^{\sigma\beta} R_{\sigma\beta}. \quad (24)$$

Using the identities above show that

$$\nabla^{\sigma} G_{\sigma\beta} = 0, \quad G_{\sigma\beta} = R_{\sigma\beta} - \frac{1}{2} \gamma_{\sigma\beta} R. \quad (25)$$

$G_{\mu\nu}$  is called the Einstein Tensor.

**17** The density of energy and momentum together forms a symmetric tensor  $T_{\mu\nu}$ . In Minkowski space, show that the equations

$$\nabla^{\mu} T_{\mu\nu} = 0 \quad (26)$$

imply that the total energy-momentum

$$P_{\mu} = \int T_{\mu 0} d^3 x \quad (27)$$

is a conserved quantity.

**18** Einstein's equations that determine the gravitational field are

$$G_{\mu\nu} = k T_{\mu\nu}. \quad (28)$$

In the Newtonian approximation, show that this reduces to Poisson's equation. Determine the relation of the constant  $k$  to Newton's constant.