Lecture 11

1. Relativity

1.1. **It is an astonishing physical fact that the speed of light is the same for all observers.** This is not true of other waves. For example, the speed of sound measured by someone on standing on Earth is different from that measured by someone in an airplane. This is because sound is the vibration of molecules of air while light is the oscillation of electric and magnetic fields: there is no medium (ether) needed for its propagation. If an observer is moving with velocity v_1 relative to air, the speed of sound measured by him/her would be

$$v = v_1 + c_s$$

where c_s is the speed of sound measured by a static observer.

1.2. **The law of addition of velocities has to be modified to take account of this fact.** The usual law (Galeleo) for addition of velocities (considering only one component for simplicty)

$$u + v$$

would lead to velocities greater than or less than c , the speed of light. The correct law is

$$\frac{u+v}{1+\frac{uv}{c^2}}.$$

1.3. **Rapidity is a more convenient variable than velocity in relativistic mechanics.** If either u or v is of magntitude c, the sum is also of magnitude c. If we make the change of variables (considering only one component for simplicity)

$$v = c\tanh\theta$$

this formula becomes simple addition:

$$\tanh[\theta_1+\theta_2] = \frac{\tanh\theta_1+\tanh\theta_2}{1+\tanh\theta_1\tanh\theta_2}.$$

The variable θ is called *rapidity.* Although velocities cannot exceed c,rapidity can be as big as you want. As $\theta \to \infty$, $v \to c$. This also suggests that a boost (change of velocity) is some kind of rotation, through an imaginary angle. Remember that $\tan i\theta = i\tanh\theta$.

1.4. **The wavefront of light has the same shape for all observers.** Imagine that you turn on and off quickly a light bulb at $t = 0$,at the position $\mathbf{x} = 0$. Light propagates along cone

$$c^2t^2 - x_1^2 - x_2^2 - x_3^2 = 0,$$

1.5. **The laws of physics must be the same for all observers.** This is the principle of relativity. It is not a discovery of Einstein: what is new is really how he reconciled this principle with the fact that the velocity of light is the same for all observers. Einstein realized that Newton's laws of mechanics need to be modified to fit with the new law for the addition of velocities. Minkowski realized that the theory of relativity can be understood geometrically: it says that the square of the distance between two events (points in space-time) is

$$c^2(t-t')^2 - (x_1 - x_1')^2 - (x_2 - x_2')^2 - (x_3 - x_3')^2$$

Lorentz transformations (changes of velocities) are like rotations in the $x - t$ plane. Because of the relative sign difference between the time and space components, these rotations are through an imaginary angle; this angle is rapidity. More generally

1.6. **The Minkowski inner product of four-vectors is.**

$$u \cdot v = u_0v_0 - u_1v_1 - u_2v_2 - u_3v_3$$

It is useful to write this as

$$u \cdot v = u^T \eta v$$

where

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

is called the *Minkowski metric.*

Remark 1. Be warned that all physicists except particle physicists, use the opposite convention, with three positive and one negative sign. Even some particle physicists use the opposite convention (e.g., S. Weinberg).

The points of space time are four vectors with components ct, x_1, x_2, x_3.

1.7. **Rapidity can be thought of as an angle in the $x\ t$ plane.** Again consider just one direction for space. Supose $x\ t$ and $x'\ t'$ are space and time as measured by two observers. Since they must agree that the velocity of light is the same, the shape of the wave front must be the same as well:

$$c^2t^2 - x^2 = c^2t'^2 - x'^2$$

This means that

$$x' = \cosh\theta x - \sinh\theta ct$$

$$ct' = \cosh\theta ct - \sinh\theta x$$

for some real number θ. Similar to the way a rotation in the plane leaves the square of the distance $x^2 + y^2$ unchanged. We can identify this angle with rapidity. Along $x' = 0$ (the position of one of the observers) the relation between $x$and t is

$$\frac{x}{t} = c\tanh\theta.$$

In other words $v = c\tanh\theta$. The above is an example of a Lorentz transformation. More generally,

1.8. **A Lorentz transformation is a 4×4 matrix that leaves the Minkowski distance unchanged.** Thus a Lorentz transformations is much like a rotation, except that the matrices must satisfy the condition

$$\Lambda^T\eta\Lambda = \eta.$$

1.9. **The products and inverses of Lorentz transformations are also Lorentz transformations.** This means that the set of Lorentz transformations forms a *group*. It is denoted by $O(1,3)$: orthogonal matrices with respect to a metric $\eta$with 1 positive sign and three negative signs.

1.9.1. *O(3) is contained as a special case of O(1,3).* A matrix that does not mix space and time

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix}$$

is a Lorentz transformation if R is orthogonal.

1.9.2. *We can split Lorentz transformations into four types using the signs of* $\det\Lambda$ *and* Λ_{00}.. If $\Lambda_{00} < 0$ the transformation includes a time reversal. If $\Lambda_{00} > 0$ and $\det\Lambda < 0$, it includes a space reversal (Parity). The subgroup with

$$\det L > 0, \quad \Lambda_{00} > 0$$

is called the set of *proper Lorentz transformations* $SO(1,3)$. Although at first all four types of Lorentz transformations appeared to be at he same footing, it turned out neither Parity nor time reversal is an exact symmetry of nature.

1.9.3. *Only the proper Lorentz transformtions are exact symmeties of nature.* Parity is violated by the weak interactions: essentially the fact that neutrinos are left handed. Time reversal is violated by the phase in the mass matrix (Kobayashi-Mazkawa matrix) of quarks.

1.9.4. *Infinitesimal Lorentz transformations form a six dimensional Lie algebra.* If $\Lambda = 1+\lambda$ for some small matrix λ, the Lorentz condition becomes

$$\lambda^T\eta + \eta\lambda = 0$$

This means that $\eta\lambda$ is anti-symmetric. A four by four anti-symmetric matrix has six independent components. Three of them represent infinitesimal rotations. The remaining three are boosts (changes of velocity) in the three co-ordinate directions. Any infinitesimal Lorentz transformation can be written as the sum of the six independent matrices:

$$L_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad L_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad L_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

(1.1)

$$L_{01} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad L_{02} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad L_{03} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

They satisfy the commutation relations

$$[L_{12}, L_{23}] = -L_{13}$$

$$[L_{12}, L_{01}] = L_{02}$$

$$[L_{01}, L_{02}] = -L_{12}$$

and their cyclic permutations. These commutation relations define the Lorentz Lie algebra. They can also be written as

$$[L_{\mu\nu}, L_{\rho\sigma}] = \eta_{\nu\rho}L_{\mu\sigma} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho} + \eta_{\mu\sigma}L_{\nu\rho}$$

1.9.5. *In addition to Lorentz transformations, translations are also symmetries. Together they form a ten dimensional algebra of symmetries, the Poincare' algebra.*

$$[L_{\mu\nu}, L_{\rho\sigma}] = \eta_{\nu\rho}L_{\mu\sigma} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho} + \eta_{\mu\sigma}L_{\nu\rho}$$

$$[L_{\mu\nu}, P_\rho] = \eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu$$

$$[P_\mu, P_\nu] = 0. \tag{1.2}$$

1.10. **Invariance under translations leads to the conservation of energy and momentum.**

1.11. **Energy and momentum transform together as a four-vector under Lorentz transformations.**

$$p = (E, cp_1, cp_2, cp_3).$$

1.12. **The relation between energy and momentum is.**

$$p \cdot p = m^2c^4, \quad E > 0$$

Geometically, this one sheet of a hyperboloid in four-dimensional space, called the *mass shell*.

$$E = \sqrt{m^2c^4 + c^2\mathbf{p}^2}$$

In particular, even a particle at rest has energy

$$E = mc^2.$$

For velocities small compared to c,

$$E \approx mc^2 + \frac{\mathrm{p}^2}{2m}$$

The second term is the Newtonian formula for kinetic energy. Since the mass of particles usually do not change,in most situations we do can ignore the first term. But in nuclear reactions, this energy can be released with spectacular results.

1.13. **For massless particles the momentum vector is null.** The inner product of momentum with itself is zero

$$p \cdot p = 0.$$

Geometrically, the set of null momenta is a cone in four-dimensional space.

The relation of energy to momentum is

$$E = c|\mathbf{p}|$$

1.14. **Free particles move along straight lines in Minkowski space.** Massive particles move along time-like straight lines: the tangent vector has positive inner product with itself. Massless particles move along null lines.

1.15. **Conservation of energy-momentum places important restrictions on decays and scattering of elementary particles.**

2. The Schrodinger equation has to be changed to take account of relativity

Recall that in quantum mechanics

$$\mathbf{p} = -i\hbar \frac{\partial}{\partial \mathbf{x}}, \quad E = i\hbar \frac{\partial}{\partial t}.$$

The relation

$$E = \frac{\mathbf{p}^2}{2m}$$

of non-relativistic mechanics gives the Schrodinger equation for a free particle:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial \mathbf{x}^2}.$$

For a relativistic particle instead

$$E^2 = c^2 \mathbf{p}^2 + m^2 c^4$$

leading to

2.1. **The Klein Gordon Equation.**

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial \mathbf{x}^2} - \left(\frac{mc}{\hbar}\right)^2 \psi.$$

The quantity $\frac{\hbar}{mc}$ has the dimension of length; it is a fundamental property of a particle determined by its mass, called its Compton wavelength. It is called that because this combination first appeared in Compton's explanation of the scattering of gamma rays by elecrons. Now we know that this equation only describes spin zero particles. Dirac discovered the correct equation for spin one half particles like the electron.

2.2. **For a massless particle this becomes the wave equation.**

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial \mathbf{x}^2}$$

2.3. **Mass and spin define the transformation properties of an elementary particle.** Mass is defined by the norm of the momentum four-vector.

$$p \cdot p = m^2$$

Spin s is defined in a similar way by

$$W \cdot W = m^2 s(s+1)$$

where

$$W^\mu = \varepsilon^{\mu\nu\rho\sigma} p_\nu L_{\rho\sigma}$$

is the Pauli-Lubanski vector. Its time component is the dot-product of angular momentum and momentum. Thus it picks out the intrisic or spin component of angular momentum: orbital angular momentum has zero dot product with momentum.

2.4. **In the quantum theory, infinitesimal translations and Lorentz transformations are represented by hermitean operators on the Hilbert space.**

2.5. **An elementary particle is such an irreducible representation of the Poincare Lie algebra.** Irreducible means that every state in the Hilbert space can be turned into any other state by some Poincare transformation. If the representation is not irreducible, there would be some subset of states that only mix with each other (form an invariant subspace) and then the system can be broken up into two pieces (the invariant subspace and its complement). So it would not be elementary or indivisible.This mathematical realization of the physical concept of an elementary particle is due to Wigner.

3. A Digression Into de Sitter Geometry

3.1. **Gravity modifies the geometry of space-time.** Particles move along geodesics (lines of least length) instead of straight lines. The departure of the geomery from Minkowski space manifests itself as the bending of the paths of particles (and light) by gravity.

3.2. **Far away from all sources of gravity, space-time will tend to Minkowski space.**

3.3. **Recent observations suggest that even afar away from all galaxies, space-time is curved.** The source of gravity at these distances is not matter: it is a mysterious new entity we call dark energy. It could well be a positive cosmological constant. Asymptotically (far away from all matter) space-time appears to have de Sitter rather than Minkowski geomery.

3.4. **de Sitter space can be thought of as the hyperboloid (pseudo-sphere).**

$$y_0^2 - y_1^2 - y_2^2 - y_3^2 - y_4^2 = -\rho^2$$

Just the sphere of radius r can be thought of as the set of points in three dimensional Euclidean space satisfying the condition

$$x_1^2 + x_2^2 + x_3^2 = r^2.$$

The extra co-ordinate does not necessarily have a physical meaning: it is only the directions within the four-dimensional subspace that describe space-time. In the limit $r \to \infty$ the sphere tends to the plane; in the limit $\rho \to \infty$ de Sitter space tends to Minkowski space. The distance ρ is several billion light years. Therefore for small distances of the size of elementary particles, we can treat space-time as if it has Minkowski geometry. Still, it is possible that the dark energy that supports de Sitter geometry has its origin in the vacuum energy of elementary particles.

3.5. **The symmetry of de Sitter space is $SO(1,4)$.** In the limit $\rho \to \infty$ it reduces to Lorentz transformations and translations. The basis of infinitesimal de Sitter transformations can be thought of as 10 five by five matrices analogous to (1.1) above. Momentum and energy are no longer conserved, as translations are not any more a symmetry of space. Still, an elementary particle would still be described by two numbers analogous to mass and spin:

$$L_{ab}L^{ab}, \quad W_a W^a$$

where

$$W^a = \varepsilon^{abcde} L_{bc} L_{de}.$$

One important consequence is the energy will not be positive. Polyakov has suggested that this makes de Sitter space unstable with respect to decay by creation of particle-antiparticle pairs. This could explain why the cosmological constant is so small: most of it is removed by the decay process. Why a tiny amount of dark energyremains would still be a mystery. Explaining dark energy is one of the most important problems in theoretical physics today.

3.6. **The de Sitter metric is.**

$$ds^2 = dt^2 - e^{-2t}(dx_1^2 + dx_2^2 + dx_3^2)$$

3.6.1. *The vectors* $e_0 = \frac{\partial}{\partial t}, \quad e_i = e^t \frac{\partial}{\partial x^i}$ *satisfy.*

$$[e_0, e_i] = e_i, \quad [e_i, e_j] = 0$$

This is a solvable Lie algebra; the de Sitter space is its Lie group. The metric is induced by the quadratic form $e_0^2 - e_1^2 - e_2^2 - e_3^3$ through left-translations. In this point of view, energy and momentum do not commute unlike in Minkowski space. Instead, energy generates scale transformations in momentum.