Lecture 12

1. MAXWELL'S EQUATIONS

The book by Jackson on *Classical Electrodynamics* has become a standard reference. The second volume of the series by Landau and Lifshitz *Classical Theory of Fields* shows greater physical insight.

1.1. All magnetic fields must have zero divergence.

$$\nabla \cdot \mathbf{B} = 0$$

This means in particular that there is no analogue to an isolated electric charge in magnetism: a permanent magnet has to be a dipole. If you cut a dipole into two we will not get an isolated North pole and South pole. Instead we will get two dipoles again. Some theories that go beyond the standard model do allow for magnetic monopoles; but none have yet been observed.

1.2. This equation can be solved by postulating that the magnetic field is a curl of a vector potential.

$$\mathbf{B} = \nabla \times \mathbf{A}$$

1.3. Two vector potentials that differ only by the gradient of a scalar give the same magnetic field.

This is called a gauge transformation

$$\mathbf{A}' = \mathbf{A} + \nabla\Lambda, \quad \mathbf{B}' = \mathbf{B}$$

$$\nabla \times \nabla\Lambda = 0.$$

It turns out that invariance under this transfomation is a fundamental symmetry of nature. We will see that gauge transformations that generalize this are the fundametnal symmetries of the standard model.

1.4. Another equation of Maxwell relates the time derivative of the magnetic field to the eletric field.

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E}$$

1.5. We can solve this by postulating in addition a scalar potential V.

$$\mathbf{E} = \frac{\partial \mathbf{A}}{\partial t} - \nabla V$$

Remark 1. Recall that we are using units such that $c = 1$. Otherwise there will be some factors of $c$all over the place.

The gauge transformations must now change the scalar potential as well

$$V' = V + \frac{\partial \Lambda}{\partial t}$$

so that the electric field is unchanged.

$$\frac{\partial \nabla \Lambda}{\partial t} = \nabla \frac{\partial \Lambda}{\partial t}.$$

1.6. **Under Lorentz transformations the scalar and vector potentials combine into a four-vector $A = (V, \mathbf{A})$.** We will introduce an index $\mu = 0, 1, 2, 3$ such that

$$A_0 = V, \quad A = (A_0, A_1, A_2, A_3)$$

Then the gauge transformation can be written as

$$A'_\mu = A_\mu + \partial_\mu \Lambda$$

where ∂_μ denotes differentiation along the μ th direction. Gauge invariance is based on the identity

$$\partial_\mu \partial_\nu \Lambda = \partial_\nu \partial_\mu \Lambda.$$

The electric and magnetic fields are then

$$E_i = \partial_0 A_i - \partial_i A_0, \quad i = 1, 2, 3.$$

$$B_1 = \partial_2 A_3 - \partial_3 A_2, \quad B_2 = \partial_3 A_1 - \partial_1 A_3, \quad B_3 = \partial_1 A_2 - \partial_2 A_1$$

This suggests that we combine them into a single matrix $F_{\mu\nu}$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

It is an anti-symmetric matrix:

$$F = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

1.7. **Scalar Products of vectors are Lorentz invariant.** Recall that there is also a symmetric matrix $\eta^{\mu\nu}$ that allows us to take products of vectors. Its indices are written above as a way of keeping track of them:

$$\eta^{\mu\nu} p_\mu q_\nu = p_0 q_0 - p_1 q_1 - p_2 q_2 - p_3 q_3, \quad \eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

It is also useful to think of the combination

$$\eta^{\mu\nu} p_\mu = p^\nu, \quad p^\nu = (p_0, -p_1, -p_2, -p_3)$$

as a vector with indices above and the scalar product as

$$p^\nu q_\nu = p_0 q_0 - p_1 q_1 - p_2 q_2 - p_3 q_3$$

A pair of indices that are repeated are summed over.

$$p^\nu q_\nu = \sum_\nu p^\nu q_\nu.$$

This convention due to Einstein simplifies the appearance of equations. It means that you must be careful not to use the same index more than twice.

1.8. **The remaining Maxwell's equations can be written in Lorentz invariant form as.**

$$\partial^\mu F_{\mu\nu} = j_\nu$$

Expanded in terms of three dimensional quantities

$$\frac{\partial \mathbf{E}}{\partial t} = -\nabla \times \mathbf{B} + \mathbf{j}$$

$$\nabla \cdot \mathbf{E} = j_0$$

The scalar j_0 is proportional to charge density and the vector $\mathbf{j}$ to current density.

1.9. **The potential A satisfies a wave equation.**

1.10. **The electromagnetic field describes a particle of mass zero and spin one.** Mass zero because it travels at the velocity of light. (Duh. it is light.) Spin one because in three dimensional language it includes a vector field, which has spin one.

2. THE DIRAC EQUATION

We saw earlier that a free spin zero massive particle is described by the Klein-Gordon equation

$$\partial^\mu \partial_\mu \phi + m^2 \phi = 0.$$

It was Dirac who discovered the correct relativistic wave equation for spin $\frac{1}{2}$ particles. Recall that such a particle has angular momentum even when it is at rest, given by the Pauli matrices $\frac{1}{2}\sigma$. Also, their wave function is not a single complex number but a pair of complex numbers. Since the spin matrices act on such a pair, it is called a *spinor*. Since both the spin and the momentum are vectors a combination such as

$$\sigma \cdot \mathbf{p} = \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix}$$

is a scalar under rotations. But it changes sign under parity (reflection of all three spatial co-ordinates). Note that

$$(\sigma \cdot \mathbf{p})^2 = |\mathbf{p}|^{\mathbf{2}}.$$

Moreover,

$$\det \sigma \cdot \mathbf{p} = -|\mathbf{p}|^2.$$

Is there a way to generalize this to get something invariant under Lorentz transformations?

2.1. The set of four matrices $(1, \sigma_1, \sigma_2, \sigma_3) = \sigma^\mu$ transform as a vector under Lorentz transformations.

2.1.1. $\sigma \cdot p == \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix}$ *is Lorentz invariant.* Note that

$$\det \sigma \cdot p = p_0^2 - |\mathbf{p}|^2 = p \cdot p$$

is the Minkowski scalar product.Using the usual quantum mechanical rule momentum can be thought of as differentiation

$$p_\mu = -i\hbar \partial_\mu$$

leading to a wave equation origially discovered (but not published) by Pauli.

Remark 2. We will use units such that $\hbar = 1$so that it will not usually appear explicitly. I put in in here just for clarity.

2.2. **The Pauli Wave Equation.**

$$\sigma \cdot \partial \phi = 0$$

Or expanded out,

$$\frac{\partial \phi_1}{\partial t} + \frac{\partial \phi_1}{\partial z} + \frac{\partial \phi_2}{\partial x} - i\frac{\partial \phi_2}{\partial y} = 0$$

$$\frac{\partial \phi_2}{\partial t} - \frac{\partial \phi_2}{\partial z} + \frac{\partial \phi_1}{\partial x} + i\frac{\partial \phi_1}{\partial y} = 0$$

2.2.1. *This describes a massless spin $\frac{1}{2}$ particle: each component of the spinor satisfies the wave equation.* The equation is Lorentz invariant, but is not invariant under parity. This because the combination $E - \sigma \cdot \mathbf{p}$ is invariant under rotation and Lorentz boosts. Under a reflection momentum changes sign but not angular momentum. (Remember that spin $\frac{1}{2}\sigma$ and orbital angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ must transform the same way). Therefore $\sigma \cdot \mathbf{p}$ changes sign. But not E.

Pauli thought this equation could describe the neutrino but rejected it because it violated parity. But in 1957 it was discovered that Parity symmetry is broken, and precisely in beta decays: that involve the neutrino!

2.2.2. *We can get a Parity invariant equation by putting together two Pauli spinors: under parity we just exchange them.*

$$(\partial_0 + \sigma \cdot \nabla)\, \lambda = 0$$

$$(\partial_0 - \sigma \cdot \nabla)\, \rho = 0$$

These still decsribe a pair of massless spin one half particles. We can get a parity invariant equation for a massive spin one half particle by making mass mix the two components:

2.3. **The Dirac equation is.**

$$(\partial_0 + \sigma \cdot \nabla)\, \lambda = im\rho$$

$$(\partial_0 - \sigma \cdot \nabla)\, \rho = im\lambda$$

The factor of i is chosen such that

2.3.1. *Each component satisfies the wave equation for massive particles.*

$$\left(\partial_0^2 - \nabla^2\right)\lambda = -m^2\lambda$$

$$\left(\partial_0^2 - \nabla^2\right)\rho = -m^2\rho$$

Thus a Dirac spinor has four components, which can be broken up into two Pauli spinors.

$$\psi = \begin{pmatrix} \lambda \\ \rho \end{pmatrix}$$

$$\begin{pmatrix} 0 & \partial_0 - \sigma\cdot\nabla \\ \partial_0 + \sigma\cdot\nabla & 0 \end{pmatrix}\psi = im\psi$$

2.3.2. *The Dirac equation can also be written as.*

$$\gamma^\mu \partial_\mu \psi = im\psi$$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\gamma = \begin{pmatrix} 0 & -\sigma \\ \sigma & 0 \end{pmatrix}$$

Recall that the Pauli matrices satisfy (indeed are defined by) the conditions

$$\sigma^i\sigma^j + \sigma^j\sigma^i = 2\delta^{ij}.$$

That is, each Pauli matrix has square one; and they ani-commute with each other. In the same spirit

2.3.3. *The Dirac matrices satisfy the conditions.*

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}.$$

Exercise. Use this identity to derive the massive wave equation for each component of the Dirac spinor, without using the explicit form in terms of Pauli matrices above.

Up to a choice of basis, all the properties of Dirac matrices follow from these conditions. In fact people use use other representations than the one above (which is called the chiral representation because it makes the parity or handedness explicit) when it suits some other purpose.

2.4. **The wave function of the electron satisfies the Dirac equation.**

2.4.1. *The muon, the tau and the six quarks all satisfy the Dirac equation, but with vastly different masses.* The masses vary from that of the electron, 0.5MeV, up to 180 GeV for the top quark. More precisely the mass is a matix whose eigenvalues have these magnitudes. It turns out that in fact the quark mass matrix has complex eigenvalues: the phase represents a violation of *CP*. This clever way of explaning *CP*violation won Kobayashi and Mazkawa a Nobel prize last year. More on all this later.

3. Quantum Electrodynamics

So far we know the equations for the wave functions of spin $0,\frac{1}{2}$ and 1 particles. To understand the interactions of these particles with each other we must introduce non-linearities. The key is gauge invariance. A complete study of the resulting theory, quantum electrodynamics is well outside the scope of this course. Itzykson and Zuber *Introduction to Quantum Field Theory* is still a good reference. At a level closer to this course is the book by Kerson Huang, *Quarks and Leptons*.

Exercise. The Dirac equation implies the conservation of a current

$$j^\mu = \bar{\psi}\gamma^\mu\psi, \quad \bar{\psi} = \begin{pmatrix} \chi^* & \phi^* \end{pmatrix}$$

That is,

$$\partial_\mu j^\mu = 0.$$

This implies that

$$\frac{\partial}{\partial t}\int j^0 d^3x = 0.$$

Thus we can think of $Q = e\int j^0 d^3x$ as the electric charge and $j^0, \mathbf{j}$ as the charge and current densities respectively. The constant e is the electric charge of the electron (or whatever other particle to which we will apply this equation).Thus

3.1. The Maxwell's equations in the presence of electrons is.

$$\partial^\mu F_{\mu\nu} = e\bar{\psi}\gamma_\mu\psi. \tag{3.1}$$

Just as electrons create electric and magnetic fields,these fields must affect their motion. The change in the Dirac equation due to the presence of electric and magnetic fields is more subtle. Gauge invariance is the key to understanding this. Recall that under gauge transformation

$$A'_\mu = A_\mu + \partial_\mu\Lambda$$

where Λ is an arbitrary function. We want to preserve this symmetry when we introuduce A_μ into the Dirac equation. We must transform ψas well so that the changes in ψ and $A_\mu$compensate for each other. Notice that if

$$\psi' = e^{ie\Lambda}\psi$$

$$\partial_\mu \psi' = e^{ie\Lambda}\left[\partial_\mu \psi + \left(ie\partial_\mu \Lambda\right)\psi\right]$$

Remark 3. Sensible people can handle the double use of the symbol e here. The e in the exponent is the electric charge and that below is the base of natural logarithms. Their values of course, have nothing to do with each other.

Thus in the combination below the derivatives of Λ cancel out:

$$\left[\partial_\mu - ieA'_\mu\right]\psi' = e^{ie\Lambda}\left[\partial_\mu - ieA_\mu\right]\psi$$

3.2. **The Dirac equation in the presence of an electromagnetic field is.**

$$\gamma^\mu\left[\partial_\mu - ieA_\mu\right]\psi = im\psi \tag{3.2}$$

Under a gauge transformation both sides are mutiplied by the same factor, so it cancels out. The pair of equations (3.1,3.2) describe Quantum ElectroDynamics (QED) of charged spin one half particles and photons.

3.3. **The equation of a charged massive spin zero particle is.**

$$\eta^{\mu\nu}\left[\partial_\mu - ieA_\mu\right]\left[\partial_\nu - ieA_\nu\right]\phi = -m^2\phi$$

This also follows using gauge invariance. Of course, here $\phi$is a scalar not a spinor.

3.4. **The proper interpretation of the equations of Quantum Electrodynamics involves renormalization.** The trouble is that the equations as describes above lead to infinities when quantum effects are fully included. They have to removed by a strange set of rules called “renormalization”. These rules work remarkably well and agree with experiments to high precision: fifteen decimal point accuracy is the best science has ever achieved. Yet the correct mathematical formulation is still not clear. Dirac himself was very unsatisfied by this situation. New ideas in analysis are needed. But that is another story.

4. LAGRANGIAN FORMALISM

4.1. **Hamilton's Variation Principle gives a concise formulation of equations of motion.** Define the Lagrangian L to be some function of position and velocity; and action to be its integral:

$$S = \int L(q, \dot{q}) dt$$

The condition that the action be stationary w.r.t. to small changes in q leads to the condition

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

With the choice

$$L = \frac{1}{2} m\dot{q}^2 - V(q)$$

this gives the Newtonian equations of motion

$$m\ddot{q} = -\frac{\partial V}{\partial q}.$$

4.2. **In a relativistic theory we the unknown quantities are fields: functions of space and time.**

4.3. **The lagrangian depends on the fields and their derivatives.** The Lagrangian is a Lorentz scalar.

4.4. **The action is the integral of the Lagrangian over space and time.**

$$S = \int L(\phi, \partial\phi) d^4x$$

$$\partial_\mu \left[\frac{\partial L}{\partial \left(\partial_\mu \phi\right)} \right] = \frac{\partial L}{\partial \phi}$$

The Lagrangian of a free massive scalar field is

$$L = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2$$

leading to the Klein-Gordon equation

$$\partial_\mu\partial^\mu\phi + m^2\phi = 0$$

More generally, an interacting scalar theory will have a lagrangian that has terms higher degree than two:

$$L = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)$$

$$\partial_\mu\partial^\mu\phi + \frac{\partial V}{\partial\phi} = 0$$

For the Higgs field of the standard model (a complex doublet)

$$L = \eta^{\mu\nu}\partial_\mu\phi^\dagger\partial_\nu\phi - V(\phi), \quad V(\phi) = \frac{\lambda}{2}\left[\phi^\dagger\phi - v^2\right]^2$$

We can see directlt that the ground states are on the sphere

$$\phi^\dagger\phi = v^2$$

4.5. **The Lagrangian of Maxwell's theory is.**

$$L = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + j^\mu A_\mu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

leading to the equation

$$\partial^\mu F_{\mu\nu} = j_\nu$$

4.6. **The Lagrangian of Dirac field is.**

$$L = \bar{\psi}\left[i\gamma^\mu\partial_\mu + m\right]\psi$$

4.7. **To get interacting theories we add the free lagrangians plus terms that depend on several fields.** For the Yukawa theory,

$$L = \bar{\psi}\left[i\gamma^\mu\partial_\mu + g\phi\right]\psi + \eta^{\mu\nu}\partial_\mu\phi^\dagger\partial_\nu\phi - V(\phi)$$

For QED

$$L = \bar{\psi}\left[i\gamma^\mu\partial_\mu + eA_\mu\right]\psi + m\bar{\psi}\psi + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

For the Abelian Higgs Model

$$L = \eta^{\mu\nu}\left[\nabla_\mu\phi\right]^*\nabla_\nu\phi - V(\phi), \quad \nabla_\mu\phi = \partial_\mu - ie\phi \quad V(\phi) = \frac{\lambda}{2}\left[\phi^*\phi - v^2\right]^2$$