Lecture 13

1. YANG-MILLS THEORY

1.1. **Yang-Mills Theory is the foundation of the theory of elementary particles.** It describes the self-interaction of spin 1 particles: the photon, $Z, W^{\pm}$ and the gluons. The principle of gauge invariance also determines the interactions of these spin one particles with those of spin zero and spin 1: the quarks and leptons.There is also a theory of interactions of spin zero particles (Higgs fields) and spin two particles (General Relativity).

1.2. **Maxwell's theory of electromagnetism is invariant under an abelian gauge group.** Let $\Lambda : R^4 \to R$ be a real valued function. Recall that under the gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \Lambda$$

the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

is unchanged. Thus the Lagrangian

$$L = \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

is invariant under both gauge and Lorentz transformations. Two successive gauge transformations is equivalent to one under the sum

$$\Lambda_1 + \Lambda_2$$

This is a commutative (abelian) group. Suppose a scalar field transforms as

$$\phi \to e^{i\Lambda} \phi$$

Then the covariant derivative

$$\nabla_\mu \phi = \partial_\mu \phi + i A_\mu \phi$$

transforms as

$$\nabla_\mu \phi \to e^{i\Lambda} \nabla_\mu \phi.$$

The Lagrangian

$$L = \frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} |\nabla \phi|^2 - V(|\phi|)$$

is gauge invariant. We saw a version of this in the discussion of the Higgs mechanism.

1.2.1. *The value of e determines the strength of the interaction.* We have chosen to define the gauge potential such that the coupling constant appears as a constant factor in the Lagrangian. For any e the gauge invariance holds. and is determined experimentally to be about a third. More precisely

$$\frac{e^2}{4\pi} \approx \frac{1}{137}.$$

1.2.2. *The commutator of covariant derivatives is just a multiplication by the field strength:*

$$\nabla_\mu \nabla_\nu \phi - \nabla_\nu \nabla_\mu \phi = iF_{\mu\nu}\phi$$

This is similar to the definition of curvature in Riemannian geometry.

1.3. **In Yang-Mills theory, the gauge transformations are valued in a Lie group.** Let $g : R^4 \to G$ be a function from space-time into a Lie group. The cases of most physical interest are $G = SU(n)$ or $U(n)$. Suppose we have a scalar field transforming under some representation of this group. (Think of $G = U(n)$ and $\phi(x) \in C^n$.) Then

$$\phi \to g\phi$$

We can define a covariant derivative by analogy

$$\nabla_\mu \phi = \partial_\mu \phi + iA_\mu \phi$$

where A_μ is valued in (matrix representation of) the Lie algebra of G. For example, if $G = U(n)$, then each component of $A_\mu(x)$ is a hermitean matrix. How should A_μ transform in order that this covariant derivative transform as before?

$$\nabla_\mu \phi \to g\nabla_\mu \phi$$

A short calculation gives the answer

$$A_\mu \to gA_\mu g^{-1} + g\partial_\mu(g^{-1})$$

If $g = e^{ie\Lambda}$ this reduces to the transformation of Maxwell's theory. What then is the analogue of the field strength? We can calculate

$$\nabla_\mu \nabla_\nu \phi - \nabla_\nu \nabla_\mu \phi = iF_{\mu\nu}\phi$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$$

The commutator term on the r.h.s. makes all the difference: it implies interactions among spin one particles that have no analogue in Maxwell's theory.

Under gauge transformations,

$$F_{\mu\nu} \to gF_{\mu\nu}g^{-1}.$$

1.3.1. *Recall that Compact Lie algebra is one that admits a positive invariant inner product.* That is, for non-zero elements of the Lie algebra

$$< u,u >> 0$$

and under the adjoint action it is invariant:

$$< gug^{-1}, gug^{-1} >=< u,u > .$$

On compact simple Lie algebras (e.g., $su(n)$) such an inner product is unique up to a scalar multiple. On $u(n) \approx su(n) \oplus u(1)$ there are two independent constants determining the general inner product. These constants are called coupling constants in the context of Yang-Mills theory

1.4. **The Lagrangian of Yang-Mills theory is determined by a positive inner product on its Lie algebra.**

$$L = \frac{1}{4} < F^{\mu\nu}, F_{\mu\nu} >$$

1.5. **Using covariant derivatives we can bring spin zero and spin one fields.**

$$L = \frac{1}{4} < F^{\mu\nu}, F_{\mu\nu} > + \frac{1}{2}|\nabla\phi|^2 + V(|\phi|)$$

$$L = \frac{1}{4} < F^{\mu\nu}, F_{\mu\nu} > + \bar{\psi}[i\gamma^{\mu}\nabla_{\mu} + m]\psi$$

1.6. **The Higgs Model with $U(2)$ invariance describes weak interactions.**

$$L = \frac{1}{4} < F^{\mu\nu}, F_{\mu\nu} > + \frac{1}{2}|\nabla\phi|^2 - V(|\phi|)$$

where $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ is a vector with two complex components. It is convenient to split the gauge field into a traceless 2×2 matrix L_μ and a multiple of the identity Y_μ since $u(2) \approx su(2) \oplus u(1)$

$$A_\mu = L_\mu + Y_\mu$$

$$L = \frac{1}{4e_1^2}\mathrm{tr}\, L^{\mu\nu}L_{\mu\nu} + \frac{1}{4e_2^2}Y^{\mu\nu}Y_{\mu\nu} + \frac{1}{2}|\nabla\phi|^2 - V(|\phi|)$$

where

$$\nabla_\mu\phi = \partial_\mu\phi + iL_\mu\phi + iqY_\mu\phi$$

The “hypercharge” q of the Higgs field and the coupling constants e_1, e_2 are experimentally determined. With the potential

$$V(\phi) = \frac{\lambda}{2}\left[\phi^\dagger \phi - v^2\right]^2$$

this describes a set of three massive particles $W^\pm, Z$ and a massless photon. (Higgs mechanism).

1.7. **Yang-Mills Theory with gauge group $SU(3)$ is Quantum Chromodynamics, the theory of strong interactions.**

$$L = \frac{1}{4\alpha}\mathrm{tr}\, F^{\mu\nu}F_{\mu\nu} + \sum_{a=1}^{N_f} \bar{\psi}[i\gamma^\mu \nabla_\mu + m_a]\psi$$

Each quark field ψ_a is a three component vector under $SU(3)$in addition to being a Dirac spinor. There are six kinds of such quarks $a = 1, \cdots 6$ corresponding to u, d, c, s, t, b with widely varying masses:

$$m_a \sim 5, 10, 1500, 250, 175000, 5000$$

in MeV. In most cases of interest in Nuclear Physics, only the lightest two or three quarks needs to be considered.

1.8. **Understanding the dynamics of non-abelian Yang-Mills theories is one of the deepest unsolved problems of theoretical physics.**