Lecture 13

1. de Sitter Geometry

1.1. **Gravity modifies the geometry of space-time.** Particles move along geodesics (lines of least length) instead of straight lines. The departure of the geomery from Minkowski space manifests itself as the bending of the paths of particles (and light) by gravity.

1.2. **Far away from all sources of gravity, space-time will tend to Minkowski space.**

1.3. **Recent observations suggest that even afar away from all galaxies, space-time is curved.** The source of gravity at these distances is not matter: it is a mysterious new entity we call dark energy. It could well be a positive cosmological constant. Asymptotically (far away from all matter) space-time appears to have de Sitter rather than Minkowski geomery.

1.4. **de Sitter space can be thought of as the hyperboloid (pseudo-sphere).**

$$y_0^2 - y_1^2 - y_2^2 - y_3^2 - y_4^2 = -\rho^2$$

Just the sphere of radius r can be thought of as the set of points in three dimensional Euclidean space satisfying the condition

$$x_1^2 + x_2^2 + x_3^2 = r^2.$$

The extra co-ordinate does not necessarily have a physical meaning: it is only the directions within the four-dimensional subspace that describe space-time. In the limit $r \to \infty$ the sphere tends to the plane; in the limit $\rho \to \infty$ de Sitter space tends to Minkowski space. The distance ρ is several billion light years. Therefore for small distances of the size of elementary particles, we can treat space-time as if it has Minkowski geometry. Still, it is possible that the dark energy that supports de Sitter geometry has its origin in the vacuum energy of elementary particles.

1.5. **The symmetry of de Sitter space is $SO(1,4)$.** In the limit $\rho \to \infty$ it reduces to Lorentz transformations and translations. The basis of infinitesimal de Sitter transformations can be thought of as 10 five by five matrices satisfying

$$[L_{ab}, L_{cd}] = \eta_{bc}L_{ad} - \eta_{ac}L_{bd} - \eta_{bd}L_{ad} + \eta_{ad}L_{bc}, \quad \eta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Momentum and energy are no longer conserved, as translations are not any more a symmetry of space. Still, an elementary particle would still be described by two numbers analogous to mass and spin:

$$L_{ab}L^{ab}, \quad W_aW^a$$

where

$$W^a = \varepsilon^{abcde}L_{bc}L_{de}.$$

One important consequence is the energy will not be positive. Polyakov has suggested that this makes de Sitter space unstable with respect to decay by creation of particle-antiparticle pairs. This could explain why the cosmological constant is so small: most of it is removed by the decay process. Why a tiny amount of dark energy remains would still be a mystery. Explaining dark energy is one of the most important problems in theoretical physics today.

1.6. **The de Sitter metric is.**

$$ds^2 = dt^2 - e^{-2t}(dx_1^2 + dx_2^2 + dx_3^2)$$

1.6.1. *The vectors* $e_0 = \frac{\partial}{\partial t}, \quad e_i = e^t\frac{\partial}{\partial x^i}$ *satisfy.*

$$[e_0, e_i] = e_i, \quad [e_i, e_j] = 0$$

This is a solvable Lie algebra; the de Sitter space is its Lie group. The metric is induced by the quadratic form $e_0^2 - e_1^2 - e_2^2 - e_3^3$ through left-translations. In this point of view, energy and momentum do not commute unlike in Minkowski space. Instead, energy generates scale transformations in momentum. This a subalgebra $e_0 = L_{04}, e_i = L_{i4}$ of the de Sitter algebra. The remaning generators $L_{\mu\nu}$for $\mu, \nu = 0, 1, 2, 3$ for the Lorentz Lie algebra. The commutators $[L_{\mu\nu}, e_\nu]$ simply say that e_ν transform as a Lorentz vector. Thus the whole difference between Minkowski and de Sitter isometries is in the commutator $[e_0, e_i]$ above.

1.7. **The Poincare algebra is a contraction of the de Sitter algebra.** If we introduce a constant ω with the dimensions of frequency

$$ds^2 = dt^2 - e^{-2\omega t}(dx_1^2 + dx_2^2 + dx_3^2)$$

the limit as $\omega \to 0$ will recover the Minkowski metric. The vectors

$$e_0 = \frac{1}{\omega}\frac{\partial}{\partial t}, \quad e_i = e^t\frac{\partial}{\partial x^i}$$

now satisfy

$$[e_0, e_i] = \omega e_i, \quad [e_i, e_j] = 0$$

In the limit $\omega \to 0$ we recover the Poincare Lie algebra. This is similar to the way that the Euclidean Lie algebra (isometry of the plane) arises as a contraction of the rotation algebra (isometry of the sphere). The paremeter ω is very small indeed: the inverse of the lifetime of the universe.