Lecture 2

1. GROUPS

Once you understand the basic structure of a physical or mathematical theory, it is useful to summarize the basic laws as axioms: independent facts from which all others can be derived. This was first achieved for plane geometry by Euclid. For mechanics by Newton. There is always a period of experimentation and discovery before a subject become mature enough to be axiomatized.Algebric concepts we study in this lectures were developed over the nineteenth century and formalized in the early twentieth century.

1.1. A *group* is a set G on which a binary operation $G \times G \to G$, is defined such that.

- $(ab)c = a(bc)$ Associative
- There is an identity element $e \in G$ such that $ea = ae = a$
- For every $a \in G$ there is an inverse such that $aa^{-1} = a^{-1}a = e$.

A group is a set of operations or transformations on some other set of objects. For example, rotations of a triangle around its centroid. Or the permutations of a deck of cards.

1.2. A subset of a group which is closed under multiplication and inverse is a subgroup.

1.2.1. *Examples.*

(1) The smallest group is the singleton set containing just one element. Its product with itself is itself, and it is its own inverse. This is called the trivial group. The empty set cannot be a group.
(2) The only set of two elements is $Z_2 = \left\{1, \omega | \omega^2 = 1\right\}$.
(3) The cyclic group of $n$elements is $Z_n = \left\{1, \omega, \omega^2, \cdots \omega^{n-1} | \omega^n = 1\right\}$. The multiplication law is $\omega^m \omega^k = \omega^{m+k}$. The inverse of ω^misω^{n-m}. It describes cyclic permutations of a set of $n$elements.
(4) The multliplication law does not have to commutative. The group of quaternions has 8 elements $1, i, j, k, -1, -i, -j, -k$ satisfying $ij = k = -ji, jk = i = -kj, ki = j = -ik, i^2 = j^2 = k^2 = -1$. Exercise: Find 2x2matrices that satisfy these relations.
(5) The permutations on a set of n elements is a group, also called the symmetrc group S_n. It has $n!$elements. Cyclic permutations are a subgroup.
(6) Any permutation can be written as a product of transpositions (pairwise permutations). Even permutations (product of an even number of transpositions) form a subgroup called the alternating group A_n.

(7) The set of integers is a group under addition but not under multiplication.
(8) The set of rational numbers is a group under addition as well. The set of non-zero rational numbers is a group under multiplication as well. (Why not if we include zero?)
(9) Similarly for real and complex numbers.
(10) The set of $n \times n$complex matrices of non-zero determinant is a group. It is denoted by $GL_n(C)$or $GL(n,C)$. GL stands for "general linear".
(11) Similarly $GL_n(R)$ is the group of real matrices of non-zero determinan one.
(12) Matrices of determinant one used to be called "special" .Thus $SL(n,C) \subset GL(n,C)$ is the subgroup of complex $n \times n$ matrices of determinant one. Similarly for $SL(n,R)$.
(13) Once the determinant is equal to one, the inverse of a matrix with integer entries also has integer entries. (Prove this.) Of course the product of two matrices with integer entries is integral as well. Thus $SL(n,Z)$ is a group although $GL(n,Z)$ is not.
(14) For example, $SL(2,Z) = \left\{ \begin{pmatrix} a & b \\ d & d \end{pmatrix} \mid ad - bc = 1 \right\}$ is a group. It is called the modular group and is important in the study of doubly periodic functions.
(15) A matrix is orthogonal if its transpose is also its inverse: $gg^T = 1$. The set of real orthogonal matrices is a group, called $O(n)$.
(16) The determinant of $g \in O(n)$ has to be ± 1. (Why?) The subset of orthogonal matrices of determinant one is called $SO(n)$.we will see that it is the group of rotations.
(17) A complex matrix is said to be unitary if its inverse is its complex conjugate transpose (hermitean conjugate): $gg^\dagger = 1$. The set of unitary matrices is $U(n)$ is a group. The subset of unitary matrices of determinant one is the group $SU(n)$. Special cases such as $SU(2)$,$SU(3)$ play important roles in particle physics.
(18) There is a close relationship between $SU(2)$ and $SO(3)$.This is important in understanding the spin of an electron.
(19) Each Platonic solid (regular solid) has a finite subgroup of SO(3) as a symmetry. The simplest is the tetrahedron, whose symmetry group is the permutation of its four vertices; if you connect the centers of the faces of a Platonic solid, you get another one called its dual. The tetrahedron is self-dual; the dual of a cube is an octahedron. The icosahedron and the duodecahedron are dual to each other. Dual solids have the same symmetry group. The symmetry group of the icosahedron plays an interesting role in the solution

of the quintic equation: it is the group of even permutations of five elements. There is much more to this story.

(20) The set of twists you can do to a Rubik's cube is a group. How many elements does it have?

(21) There is a huge literature on applying symmetry groups to molecular and crystal physics. This used to be the main applicaion of groups to physics. We won't do much of that in this course. All physics is either particle physics or stamp collecting. (Not really).

(22) Many viruses have symmetric shapes. It is not yet clear if this is important in understanding their biology.

1.3. **A map from a group to another that preserves multiplication is called a homomorphism. An isomorphism is a homomorphism that is one-to-one and onto.** Thus a homomorphism $f : G \to H$ will satisfy

$$f(g_1g_2) = f(g_1)f(g_2)$$

The set of elements of G that are mapped to the identity of H is called the *kernel* of this homomorphism. The kernel of any homomorphism is a subgroup. If there is an isomorphism between two groups they have the same structure: at some abstract level they are identical. Of course,the kernel of an isomorphism is trivial.

1.3.1. *Examples.*

(1) There is an isomorphism from the cyclic group to the group of nth roots of unity: $\omega \mapsto e^{\frac{2\pi i}{n}}$.
(2) The determinant is a homomorphism from $O(n) \to \{1,-1\}$. Its kernel is $SO(n)$.
(3) There is a homomorphism $h : SU(2) \to SO(3)$ whose kernel is the subset $\{1,-1\}$. We will study this one in more detail later as it is important in quantum mechanics.

2. Representations

2.1. **A homomorphism** $r : G \to GL(n,C)$ **is called a representation.** Representations allow us to think of group elements in terms of matrices, which are much more concrete objects almost as familiar as numbers.

2.1.1. *A unitary representation is a homomorphism* $r : G \to U(n)$;*an orthogonal representation is* $r : G \to O(n)$ *etc.* In quantum mechanics, symmetries are unitary representations. This are therefore the most important representations. In general the representations might be in terms of infinite dimensional matrices. But we will mostly confine ourselves to finite dimensional matrices as the theory is so much simpler, but still useful in physics.

Unless we say otherwise, all representations we study will be unitary or orthogonal.

2.1.2. *Example.*

(1) Any element of the permutation group S_3 can be written as a product of a cyclic permutation $\omega \equiv ABC \to BCA$ and a reflection $\tau \equiv ABC \to ACB$. We say that S_3 is generated by ω, τ . A representation in $O(3)$ is given by

$$\omega \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

2.1.3. *Recall that the direct sum of two matrices is given by stacking them as a bigger matrix* $A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. *This can be extended to the direct sum of two representations* $r_1 \oplus r_2(g) = \begin{pmatrix} r_1(g) & 0 \\ 0 & r_2(g) \end{pmatrix}$.. Clearly, the direct sum of unitary representations is also unitary. Physically, the direct sum of representations describes two sets of states that are not mixed into each other by the action of the symmery group. For example, rotations do not mix the p states of hydrogen with the $d$states. They belong to two separate direct summands in the representation of the rotation group on the space of states of hydrogen. More on this later.

2.1.4. *Conversely, if a unitary representation can be decomposed as the sum of two other representations (if every representation matrix is the direct sum of two others) we say that the representation is reducible.*

2.1.5. *Two representations are equivalent if there is an invertible matrix S (independent of g) such that* $r_1(g) = Sr_2(g)S^{-1}$.. If S is unitary we say that they are unitarily equivalent. Equivalent representations only differ by a choice of basis: not different in any essential way.

2.1.6. *The direct product (also called tensor product) of two representations is defined in terms of the direct product of matrices in a similar way.* Physically the direct product describes a system that has two parts. For example, the hydrogen atom has a proton and an electron. The representation of the rotation group is the product of the representations on each constituent.

3. Vector Spaces

It is useful to have an axiomatic point of view on vectors and matrices as well. Refer to my notes online on vector spaces for the course PHY402/404. In particular it is useful to know about tensors.

4. Review of Quantum Mechanics

A good reference is R. Shankar, *Principles of Quantum Mechanics* . The classic text remains Dirac's book of the same title. You will be a walking encyclopedia of quantum mechanics if you master the third volume of the *Course in Theoretical Physics* by Landau and Lifshitz. The summary below is not a substitute for a proper course in quantum mechanics: it usually takes a year to learn the material contained in this section.

4.1. **The Postulates.** Quantum theory is still not completely developed. Questions about measurement and interpretation are still being worked out (e.g.,"weak measurement"). Also, combining relativity with quantum mechanics (quantum field theory) leads to infinities that have to be removed by a mysterious procedure known as renormalization. No one is satisfied with this situation. Even worse, we have not yet been able to reconcile quantum mechanics with general relativity to get a quantum theory of gravity. Also, the discovery of dark energy suggests that something is seriously wrong in the way we think of the energy of the vaccuum.

Nevertheless we can say, after almost a century of experimental tests, a few things for certain about how quantum theory works. It is not too early to summarize them as a set of postulates.

4.1.1. *The states of a physical system are represented by vectors in a complex Hilbert space.* This means the we can take linear combinations

$$\alpha|\psi> + \beta|\phi>$$

of two states $|\psi>$ and $\phi>$ to get another state. The quantities α, β are complex numbers. There is a way to take the inner product (scalar product) of two states to get a complex number

$$<\psi|\phi> .$$

This inner product is linear in the second argument

$$<\psi|\alpha\phi + \beta\chi> = \alpha <\psi|\phi> + \beta <\psi|\chi>$$

and anti-linear in the first entry

$$<\alpha\psi + \beta\chi|\phi> = \alpha^* <\psi|\phi> + \beta^* <\chi|\phi> .$$

Remark 1. Be aware that mathematicians use the opposite convention: for them it is the second entry in an inner product that is anti-linear. Mathematics and physics are two neighboring cultures divided by a common language.

Moreover, the inner product of any vector with itself is positive; it is only zero for the zero vector. Thus

$$|\psi|^2 =< \psi|\psi >$$

can be thought of as the square of the length of a vector.

Remark 2. Strictly speaking states are represented by rays (directions) in Hilbert space. It is a fine point though.

A typical situation is that the state is a complex valued function of some real variable (e.g., position), The inner product is then

$$< \psi|\psi >= \int \psi^*(x)\phi(x)dx.$$

Exercise 3. Verify that this integral has the properties of an inner product.

Or, the states may be represented by a column vector with complex components $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ etc.

Exercise 4. Prove that

$$\frac{| < \psi|\phi > |^2}{|\psi|^2|\phi|^2} \leq 1$$

for all non-zero states $|\phi >, |\psi >$. This is called the Cauchy-Schwarz inequality.

4.1.2. *If a system is in state $|\phi >$, the probability of finding it in another state $|\psi >$is* $\frac{|<\psi|\phi>|^2}{|\psi|^2|\phi|^2}$.. This is one of the confusing things about quantum mechanics, until you get used to it. A classical analogue is the polarization of light. About half of circularly polarized light will pass through a filter that allows only linearly polarized light.

4.1.3. *The observables of a physical system are hermitean linear operators on the states.* A linear operator (or matrix) acting on a state produces another state, such that

$$L(\alpha|\psi > +\beta|\chi >) = \alpha L|\psi > +\beta L|\chi > .$$

Hermitean operators satisfy in addition

$$< \psi|L|\chi >=< \chi|L|\psi >^* .$$

That is, the conjugate-transpose of a matrix elements is itself. If

$$L|\psi_\lambda >= \lambda|\psi_\lambda >$$

for some complex number λ and non-zero vector $|\psi_\lambda >$, we say that $\psi_\lambda >$is an eigenvector of $L$with eigenvalue λ. The most important property of a hermitean operator is that it has real eigenvalues. That comes in handy because

4.1.4. *The possible outcomes of measuring an observable are its eigenvalues.* Even if if we know the state of a system, we may not be able to predict the outcome of measuring an observable. The best we can do is to give probabilities. With $\lambda, |\psi_\lambda >$ defined as above,

4.1.5. *If the system is in some state $|\phi >$,the probability of getting the outcome λ upon measuring L is.*

$$\frac{|< \psi_\lambda|\phi > |^2}{|\psi_\lambda|^2|\phi|^2}.$$

Exercise 5. Recall that this is always real and less than one. Why do the probabilities add up to one?

4.1.6. *There is a hermitean operator called the hamiltonian which represents energy; the time dependence of a state is given by.*

$$i\hbar\frac{\partial|\psi >}{\partial t} = H|\psi > .$$

Thus if you know the state at some time, you can in principle predict what it will be at some later time. If you know the exact hamiltonian and if it is simple enough to make the equation solvable.

4.2. **Electron in a Magnetic Field.** As an example, think of an electron in a magnetic field. It is bound to an atom (e.g., Sodium) and we ignore the change in its position: only the rotation of its spin. The wave function has two components. The energy of an eletron in a magnetic field is proportional to the dot product of the spin and the magnetic field

$$H = \mu\sigma \cdot B \tag{4.1}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices.

Exercise 6. Find the eigenvalues and eigenfunctions of the hamiltonian (4.1). If the initial state at time $t = 0$ is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the magnetic field is in along the $x-$axis $B = (B, 0, 0)$ what is the probability that a measurement of σ_3 at a time t will yield the value -1? This illustrates the phenomenon of oscillationof quantum states, also important for neutrinos.

4.3. **Symmetries and Conservation Laws.**

4.3.1. *Symmetries are represented by unitary operators that commute with the hamiltonian.* Recall that the probability of finding a particle in state ψ in another state ϕ is $| < \phi|\psi > |^2$ (assuming that the state vectors are of length one.) If the symmetry is represented by a linear transformation L it must satisfy

$$| < L\phi|L\psi > | = 1.$$

Also recall the definition of the hermitean conjugate (adjoint)

$$< L\phi|\psi >=< \phi|L^\dagger \psi >$$

Thus we see that one way to satisfy the condition is to have

$$L^\dagger L = 1$$

That is, **unitary transformations**. Most symmetries are of this type. (See below for an exception.)

Recall that a state of energy E is an eigenstate of the hamiltonian.

$$H\psi = E\psi$$

A symmetry must take it to another state of the same energy:

$$H(L\psi) = E(L\psi).$$

This is satisfied if

$$HL = LH.$$

That is, if the hamiltonian commutes with the symmetry operator. Thus a symmetry is represented by a unitary operator that commutes with the hamiltonian:

$$L^\dagger L = 1, \quad HL - LH = 0.$$

4.3.2. *An exceptional case is time reversal, which is an anti-linear operator.*

$$\Theta(a\psi + b\phi) = a^* \Theta\psi + b^* \Theta\phi.$$

We won't have much more to say about this case for now; we will only consider the case of linear operators for now.

4.3.3. *An example is Parity.* It reverses the sign of the co-ordinates of a particle

$$P\psi(x) = \psi(-x).$$

Clearly $P^2 = 1$.

The Schrödinger equation for a free particle is invariant under this transformation

$$-\frac{\hbar^2}{2m}\nabla^2\psi = -i\hbar\frac{\partial\psi}{\partial t}.$$

Another way of seeing that this is a symmetry is that the operator P commutes with the hamiltonian

$$H = -\frac{\hbar^2}{2m}\nabla^2, \quad PH = HP.$$

Thus if ψ is a state with energy E

$$H\psi = E\psi$$

so will be $P\psi$. Even with a potential parity continues to be a symmetry if

$$V(-x) = V(x).$$

For example consider a particle in one dimension with a potential

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V, \quad V(x) = \lambda(x^2 - a^2)^2, \quad \lambda > 0.$$

There are two minima at $x = \pm a$. The eigenstates of energy can also be simultaneously eigenstates of parity because $[H,P] = 0$. It turns out that the ground state is of even parity

$$\psi(-x) = \psi(x)$$

while the first excited state is of odd parity

$$\psi(-x) = -\psi(x).$$

4.3.4. *Translation invariance leads to conservation of momentum.* The translation by a is represented by the operator

$$T(a)\psi(x) = \psi(x+a).$$

A free particle on a line has hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V$$

with a constant potential. Thus whether we apply the hamiltonian before or after a translation we get the same effect on a wavefunction:

$$HT(a) = T(a)H.$$

For a particle moving in one dimension, an infinitesimal translation is represented by the derivative operator:

$$\psi(x+a) \approx \psi(x) + a\frac{\partial \psi}{\partial x} + \cdots$$

Thus if a system is invariant under translation, its hamiltonian must satisfy

$$[H, \frac{\partial}{\partial x}] = 0.$$

The operator $\frac{\partial}{\partial x}$is anti-hermitean. The corresponding hermitean operator is $-i\frac{\partial}{\partial x}$. If we multiply by $\hbar$ we get the momentum operator

$$p = -i\hbar\frac{\partial}{\partial x}.$$

Thus translation invariance implies the conservation of the momentum:

$$[H, p] = 0.$$

Similar arguments apply to each component of momentum of a free particle moving in R^3.

4.3.5. *Rotation invariance implies conservation of angular momentum.* The infinitesimal generators of rotation are

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \quad \mathbf{p} = -i\hbar\frac{\partial}{\partial \mathbf{r}}.$$

They satisfy the relations

$$[L_1, L_2] = i\hbar L_3, \quad [L_2, L_3] = i\hbar L_1, \quad [L_3, L_1] = i\hbar L_2.$$

4.3.6. *In quantum mechanics, a particle can have angular momentum even when its momentum is zero.* Total angular momentum is the sum of the orbital angular momentum and an intrinsic angular momentum

$$\mathbf{J} = \mathbf{L} + \mathbf{S}.$$

The components of S are a set of three matrices satisfying

$$[S_1, S_2] = i\hbar S_3, \quad [S_2, S_3] = i\hbar S_1, \quad [S_3, S_1] = i\hbar S_2.$$

The simplest choice is $S = 0$. There are several such spin zero particles; e.g., the alpha particle. The next simplest choice is

$$S_1 = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad S_2 = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad S_3 = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These are 'spin half' particles, since the maximum eigenvalue of a component of spin is half of $\hbar$. An electron, a proton, a neutron are all examples of such particles.

The photon has spin one. But it cannot be described by the above theory because it moves at the speed of light. We need relativistic quantum mechanics for that.

There are a set of particles called Δ that have spin $\frac{3}{2}$. Their spin is represented by 4x4 matrices. There are particles with even higher spin but they tend to be unstable.

5. ROTATIONS

5.1. **The distance between two points** $x = (x_1, x_2 x_3)$ **and** $y = (y_1, y_2, y_3)$ **is given by** $(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2$**.** If we translate both vectors by the same amount , the distance is unchanged. Similarly if we rotate both the same way, the distance is unchanged.

5.2. **The square of the length of a vector, thought of as a** 3×1 **matrix is** $x^T x$**.** A rotation is described by a 3×3 matrix R

$$x \to Rx.$$

The length of x is unchanged if

$$R^T R = 1$$

Such matrices are called *orthogonal*. Not all orthogonal matrices describe rotations though. The matrix

$$P = -1$$

reverses the sign of all three co-ordinates, which cannot be achieved by any rotation. Yet it is orthogonal. Every orthogonal matrix has determinant ± 1. Every rotation can be built up by rotations through very small angles. Since the sign of the determinant cannot suddenly jump from 1 to -1, the determinant of a rotation matrix must be the same as for the identity. That is,

5.3. **A rotation is a matrix that is both orthogonal and of determinant one.** The set of all 3×3 orthogonal matrices is called $O(3)$. Matrices of determinant one used to be called special matrices. Thus the set of rotation matrices is called $SO(3)$.

5.4. **If** $g, h \in SO(3)$, **then** $gh \in SO(3)$ **and** $g^{-1}, h^{-1} \in SO(3)$ **as well.** Thus $SO(3)$is a *group*: it is closed under multiplication of these matrices as well as taking their inverses.

5.5. **An infinitesimal transformation $R = 1 + A$ is orthogonal $A^T + A = 0$; i.e., infinitesimal rotations are described by anti-symmetric matrices.** An arbitrary anti-symmetric matrix can be written as a linear combination of the basic ones

$$S_{12} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad S_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

which describe rotations in each co-ordinate plane. They satisfy the commutation relations (verify this)

$$[S_{12}, S_{23}] = S_{13}, \quad [S_{23}, S_{13}] = S_{12}, \quad [S_{13}, S_{23}] = -S_{12}$$

This is an example of what is called a Lie algebra. It has turned out that elementary particles are classified by algebras $SU(3), SU(2)$ such as these, only more complicated. Note that the Pauli matrices satisfy very similar commutation relations.

Exercise 7. What are the commutation relations of rotations in four dimensions?