Lecture 3

1. Lie Algebras

1.1. **A Lie algebra is a vector space along with a map** $[.,.]:\mathscr{L}\times\mathscr{L}\to\mathscr{L}$ **such that,**

$$[\alpha a+\beta b,c]=\alpha[a,c]+\beta[b,c] \quad \text{bi}-\text{linear}$$

$$[a,b]=-[b,a] \quad \text{Anti}-\text{symmetry}$$

$$[[a,b],c]+[[b,c],a][[c,a],b]=0, \quad \text{Jacobi identity}$$

We will only think of real vector spaces. Even when we talk of matrices with complex numbers as entries, we will assume that only linear combinations with real combinations are taken.

1.1.1. *A homomorphism is a linear map among Lie algebras that preserves the commutation relations.*

1.1.2. *An isomorphism is a homomorphism that is invertible; that is, there is a one-one correspondence of basis vectors that preserves the commutation relations.*

1.1.3. *An homomorphism to a Lie algebra of matrices is called a representation. A representation is faithful if it is an isomorphism.*

1.2. **Examples.**

(1) The basic example is the cross-product in three dimensional Euclidean space. Recall that

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

The bilinearity and anti-symmetry are obvious; the Jacobi identity can be verified through tedious calculations. Or you can use the fact that any cross product is determined by the cross-product of the basis vectors through linearity; and verify the Jacobi identity on the basis vectors using the cross products

$$\mathbf{i}\times\mathbf{j}=\mathbf{k},\quad \mathbf{j}\times\mathbf{k}=\mathbf{i},\quad \mathbf{k}\times\mathbf{i}=\mathbf{j}$$

Under many different names, this Lie algebra appears everywhere in physics. It is the single most important example of a Lie algebra.

(2) The commutator of matrices is the other important example:

$$[A,B]=AB-BA$$

Again, anti-symmetry is obvious. The Jacobi identity follows from the associativity of matrix multiplication. We will see that this is the infinitesimal version of the group $GL(n)$.

(3) Various sub-algebras of the algebra of matrices provide the other important examples. The product of anti-symmetric matrices need not be either symmetric or antisymmeric:

$$(AB)^T=B^TA^T=BA.$$

But the commutator of anti-symmeric matrices is always anti-symmetric

$$[A,B]^T=(AB-BA)^T=BA-AB=-[A,B].$$

This Lie algebra is the infinitesimal version of the orthogonal group $O(n)$:recall that an orthogonal matrix that is infinitesimally close to the identity is of the form $1+A$ with $A^T=-A$. We will call this Lie algebra $o(n)$.

(4) Similarly, the commutator of anti-hermitean matrices is anti-hermitean. This Lie algebra $u(n)$ is the infinitesimal version of the group of unitary matrices $U(n)$.

(5) The property of being traceless is preserved under the commutator. Thus we have the Lie algebra of traceless anti-hermitean matrices $su(n)$ which is the infinitesimal version of the group $SU(n)$ of unitary matrice of determinant one. Recall that if a matrix is infinitesimemally close to one, $\det(1+A)\approx 1+\mathrm{tr}A$.

(6) The Lie algebra $o(3)$ is in fact the same as the cross-product on three dimensional vectors. For, any antisymetric matrix can be written as

$$A=A_{12}S_{12}+A_{23}S_{23}+A_{13}S_{13},$$

$$S_{12}=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix},\quad S_{23}=\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix},\quad S_{13}=\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix}$$

The commutation relations

$$[S_{23},S_{13}]=S_{12},\quad [S_{13},S_{12}]=S_{23},\quad [S_{12},S_{23}]=S_{13}$$

are isomorphic to those above under the correspondence $\mathbf{i}\mapsto S_{23},\mathbf{j}\mapsto S_{13},\mathbf{k}\mapsto S_{12}$.

(7) Moreover the Lie algebra $su(2)$ is isomorphic to $o(3)$. Any traceless anti-hermitean matrix can be written as a linear combination of Pauli matrices

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

The correspondence $S_{12} \mapsto \frac{i}{2}\sigma_3, \quad S_{23} \mapsto \frac{i}{2}\sigma_1, \quad S_{13} \mapsto \frac{i}{2}\sigma_2$ gives an isomorphism. This is fundamental to understanding the spin of an electron.

(8) The Poisson bracket of classical mechanics was the first example of a Lie algebra: this is where Lie discovered it. Recall that observables of classical mechanics are functions of positions and momenta. For a single degree of freedom (for simplicity) the Poisson bracket is defined as

$$\{A,B\} = \frac{\partial A}{\partial p}\frac{\partial B}{\partial q} - \frac{\partial A}{\partial q}\frac{\partial B}{\partial p}$$

Verifying the Jacobi identity for this is a heart-warming exercise. For more than one degree of freedom we sum over each pair of conjugate variables:

$$\{A,B\} = \sum_i \left(\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} - \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} \right)$$

(9) In particular we have the canonical commutation relations (also called the Heisenberg algebra)

$$\{p,q\} = 1, \quad \{p,1\} = \{q,1\} = 0.$$

This is an example of a nilpotent algebra: if we take repeated commutators, the brackets vanish.

(10) The Poisson brackets of the components of angular momentum provide yet another physically important realization of the Lie algebra $o(3)$

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

$$\{L_1,L_2\} = L_3, \quad \{L_2,L_3\} = L_1, \quad \{L_3,L_1\} = L_2$$

This isomorphism arises because the canonical transformations generated by angular momentum are rotations.We can regard the earlier examples in terms of matrices as representations of this Lie algebra of the angular momentum components.

(11) A Lie algebra that is commutative is trivial: the bracket must vanish. Thus, to be interesting a Lie algebra must be non-abelian.

(12) The only Lie algebra of dimension one is the trivial algebra.

(13) The only non-abelian Lie algebra of dimension two can be written as

$$[e_0,e_1] = e_1$$

by a choise of basis. Find a representation for it in terms of 2×2 matrices. Answer: $e_0 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e_1 \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

(14) There are a handful of three dimensional Lie algebras. We already saw $o(3) \approx su(2)$in its various guises as the most important of all Lie algebras; it is three dimensional because there are three independent basic elements in it. Another three dimensional Lie algebra, which is not isomorphic to this one, is called $sl(2,R)$

$$[e_1,e_2] = -e_3, [e_3,e_2] = e_1, \quad [e_3,e_1] = e_2.$$

The sign of the first commutator is what distinguishes this from $o(3)$.

(15) Find an isomorphism of $sl(2,R)$ with the space of traceless real 2×2 matrices.

(16) Also, find a set of three functions of position and momentum with Poisson brackets isomorphic to $sl(2,R)$.

(17) In addition to matrix algebras such as $su(n), so(n)$ there is also a finite sequence of exceptional Lie algebras. Many physicists have tried hard to explain various physical phenomena in terms of exceptional Lie algebras because they are so mathematically beautiful. But nothing has worked. We will stay away from them in this course.

2. Lie Groups

2.1. **A Lie group is a group in which there is a co-ordinate system such that the multiplication and inverse are differentiable functions.** In other words, a Lie group is a manifold along with a multiplication and inverse are differentiable functions. (If you don't know what a manifold is don't worry about this).

2.1.1. *A discrete group like the set of integers, or the set of of rationals, or a finite group like the permutation groups, arenot Lie: there is no way to differentiate group elements.*

2.1.2. *$GL(n,R)$ is a Lie group.* The matrix elements themselves provide a co-ordinate system. The condition $\det g \neq 0$ leaves behind an open neighborhood of R^{n^2}.

2.1.3. *$U(n)$ is a Lie group.* We have to solve the constraints defining the group

$$g^\dagger = g^{-1}$$

The matrix elements themselves are no longer a co-ordinate system: we need to solve these rather complicted nonlinear equations.The substitution

$$g = e^a$$

allows us to solve them. The exponential of a matrix is defined by an infinite series in the same way as the exponential of a number

$$e^a = 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \cdots$$

It satisfies the conditions

$$(e^a)^\dagger = e^{a^\dagger}, \quad (e^a)^{-1} = e^{-a}.$$

(Prove these.) The tricky part is that

$$e^a e^b \neq e^{a+b}$$

unless $[a,b] = 0$. There is a much more complicated formula that replaces this which we will come to soon. But just the above two identities are enough to solve the unitarity condition.

2.1.4. *If $a^\dagger = -a$, then e^a is a unitary matrix.* The matrix elements of the anti-hermitean matrix provide a co-ordinate system on $U(n)$ in the neighborhood of the identity. More precisely, define the norm of an anti-hermitean matrix by $|a|^2 = \mathrm{tr} a^\dagger a$. As long as $|a| < \pi$, the exponential function is injective (i.e., e^a completely defines a within the disc $|a| < \pi$). This establishes a co-ordinate system around the origin. Next we establish a co-ordinate system around the roots of unity by setting $g = e^{\frac{2\pi i}{n}k} e^b$, for $k = 0, 1, \cdots n-1$ again with $|b| < \pi$. It is possible to show (we will omit the detials of the construction and the proof) these n co-ordinate systems cover all over $U(n)$:

2.1.5. *Every unitary matrix can be written in the form $g = e^{\frac{2\pi i}{n}k} e^b$ for some $k = 0, 1 \cdots n-1$ and $b^\dagger = -b$with $|b| < \pi$..* Of course most parts of $U(n)$are covered by more than one of these co-ordinate systems: the changes of variables from one system to the other is differentiable. This is similar to the way that a polar co-ordinate system cannot cover all of the plane: the origin and the line $\theta = 0$ have to be excluded. But two polar systems with different centers and axes can cover all of the plane; in regions covered by both systems we can differentiably change varaibles among them

2.2. **If** $a^\dagger = -a$ **and** $\mathrm{tr} a = 0$,**then** $e^a \in SU(n)$**.** The point is that $\det e^a = e^{\mathrm{tr} a}$. This identity is obvious for matrices that can be diagonalized (Prove it!). More gnerally it follows by continuity as the determinant, trace and exponential are all continuous functions and matrices that cannot be diagonalized can be perturbed infinitesimally and made diagonalizable. This makes $SU(n)$ into a Lie group by similar arguments.

2.3. **If $a^T = -a$, then $e^a \in SO(n)$.** Recall that anti-symmetric matrices have zero trace. Hence $\det e^a = e^{\mathrm{tr} a} = 1$. It is not possible to express Parity as the exponential of an anti-symmetric matrix.

2.4. **Every Lie group determines a Lie algebra.** More than one Lie group might lead to the same Lie algebra. For example, $SU(2)$and $SO(3)$ yield the same Lie algebra, although they are not isomprphic as groups.

3. From Lie Groups to Lie Algebras

3.1. **The set of elements infinitesimally close to the identity in a Lie group form a Lie algebra.** For matrix groups like $GL(n,R), SU(n), SO(n)$above we put

$$g = e^a, \quad h = e^b$$

with a, b having infinitesimally small parameters, we can compute

$$e^a \approx 1 + a + \frac{a^2}{2} + O(a^3)$$

$$g^{-1} = e^{-a} \approx 1 - a + \frac{a^2}{2} + O(a^3)$$

$$gh \approx 1 + a + b + \frac{a^2 + 2ab + b^2}{2} + O(a^3, b^3)$$

$$g^{-1}h^{-1} = 1 - a - b + \frac{a^2 + 2ab + b^2}{2} + O(a^3, b^3)$$

$$ghg^{-1}h^{-1} = 1 + [a,b] + O(a^3, b^3)$$

(Calculate each line out and verify this.) Thus the lack of commutativity of group multiplication taken to second order determines the commutator.

3.2. **The Lie algebra of $U(n)$ is $u(n)$,the set of anti-hermitean matrices; that of $SU(n)$ is $su(n)$, the traceless anti-hermitean matrices.**

3.3. **The Lie algebra of $SO(n)$ is $o(n)$ the set of anti-symmetric matrices.**

4. From Lie Algebras to Lie Groups

The passage from Lie groups to Lie algebras is a kind of differentiation. The converse is a kind of non-commutative integration: you should expect this to be much harder. It is actually beyond the scope of this course. The material below is just a guide to those who want to venture further. The book by M. Hausner and J. T. Schwartz *Lie Groups and Lie Algebras* is quite good for this topic. A purely algebraic proof of the BCH lemma is in the book *Free Lie Algebras* by Reuttenauer.

4.1. **The Lie bracket completely determines the group multiplication.** In the exponential co-ordinate system, the multiplication of group elements follows from taking repeated commutators and adding them up in a particular way. The key is a formula that allows us to multiply the exponentials of matrices that do not commute, called the

4.2. **The Baker-Campbell-Hausdorff Formula.**

$$e^a e^b = e^{a+\int_0^1 dt\,\psi\left(e^{\hat{a}}e^{t\hat{b}}\right)b}$$

Here

$$\psi(x) = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \equiv \frac{x\log x}{x-1}$$

is the generating function for Bernoulli numbers. We denote the operation of taking a commutator by $\hat{a}b \equiv [a,b]$ (this is also called ad$a$ in mathematics books) so that $\hat{a}^2 b \equiv [a,[a,b]]$,$\hat{a}^3 b = [a,[a,[a,b]]]$ etc.

For the first few terms

$$e^a e^b = e^{a+b+\frac{1}{2}[a,b]+\frac{1}{12}([a,[a,b]]+[b,[b,a])-\frac{1}{24}[b,[a,[a,b]]]+\cdots}$$

To prove this we will need a series of intermediate results

Lemma 1.

$$e^a b a^{-a} = e^{\hat{a}} b \equiv b + [a,b] + \frac{1}{2!}[a,[a,b]] + \frac{1}{3!}[a,[a,[a,b]]] + \cdots$$

Proof. Let $b(t) = e^{ta} b e^{-ta}$,$b(0) = b$.Then

$$b(t+\varepsilon) = e^{(t+\varepsilon)a} b e^{-(t+\varepsilon)a} = e^{\varepsilon a} e^{ta} b e^{-ta} e^{-\varepsilon a} \approx (1+\varepsilon a) b(t)(1-\varepsilon a) \approx b(t) + \varepsilon [a, b(t)]$$

up to terms second order in ε.Thus

$$\frac{d}{dt} b(t) = [a, b(t)] = \hat{a} b(t).$$

Regarding $\hat{a}$ as a linear operator on b,the solution is

$$b(t) = e^{t\hat{a}} b.$$

$\square$

Lemma 2. *Let $a(t)$ be a function of a real variable valued in the Lie algebra. Then , with $\phi(z) = \frac{e^z-1}{z} = 1 + \frac{1}{2!}z + \frac{1}{3!}z^2 + \cdots$*

,

$$\frac{d}{dt}e^{a(t)} = \phi(-\hat{a}(t))\frac{da(t)}{dt}.$$

Proof. Define $g(s,t) = e^{sa(t)}$. Then

$$A(s,t) = g^{-1}\frac{\partial g}{\partial s} = a(t)$$

by the definition of the exponential. Define

$$B(s,t) = g^{-1}\frac{\partial g}{\partial t} = e^{-sa(t)}\frac{\partial e^{sa(t)}}{\partial s}.$$

We can verify the identity

$$\frac{\partial B}{\partial s} - \frac{\partial A}{\partial t} + [A,B] = 0$$

which now becomes

$$\frac{\partial B}{\partial s} - \frac{\partial a(t)}{\partial t} + [a(t),B] = 0$$

or

$$\frac{\partial B}{\partial s} = -\widehat{a(t)}B + \dot{a}, \quad B(0,t) = 0$$

The dot denotes differentiation w.r.t. t.

We can think of $t$as a constant and solve this as a power series in s:

$$B(s,t) = s\dot{a} + \frac{s^2}{2!}(-\widehat{a(t)})\dot{a} + \cdots \frac{s^n}{n!}\left(-\widehat{a(t)}\right)^{n-1}\dot{a}$$

Putting $s = 1$in this we get the result we want. $\square$

Lemma 3. *Let $e^a e^{tb} = e^{c(t)}$.Then*

$$e^{-c(t)}\frac{d}{dt}e^{c(t)} = b$$

Proof. Just calculate:

$$\frac{d}{dt}e^{c(t)} = e^a\frac{d}{dt}e^{tb} = e^a e^{tb} b = e^{c(t)}b.$$

$\square$

Now we can prove the BCH formula

Proof. Using the Lemmas 2,3 above

$$\phi(-\widehat{c(t)})\dot{c} = b$$

Now the function $\psi(z) = \frac{z\log z}{z-1}$ satisfies

$$\psi(z)\phi(-\log z) = 1$$

so that

$$\frac{dc}{dt} = \psi(e^{\widehat{c(t)}})b$$

If we integrate this differential equation (recall the boundary condition $c(0) = a$)and evaluate it at $t = 1$, we get the result claimed. □

All this leads up to a fundamental theorem of Lie.

4.3. **A Lie algebra determines a unique simply connected Lie group. Every Lie group with this Lie algebra is a quotient of this simply connected Lie group by a countable abelian group.** To understand this result you have to know a bit of algebaic topology: a simply connected space. The most important physical application is to spin which we will discuss in detail later.