Lecture 5

1 The Rigid Body

1.1 If the distance between any two points of a body is fixed as it moves, it is a rigid body.

Thus a rigid body can move by a translation of its center of mass; and a rotation around its center of mass. Imagine throwing a book into the air. Its center of mass will follow a parabolic trajectory as it turns around its axes. The complicated part of the story is the rotation, and we will focus on it exclusively.

1.2 We will study in this section only the case when the total torque on the body is zero.

This will be complicated enough. The angular momentum $\mathbf{R}$ of the body as measured by an inertial observer is conserved in this case.

1.3 The angular velocity of a body is the vector pointing along the axis of rotation and having as magnitude the rate of change of the angle.

1.3.1 The velocity at any point of a rigid body is

$$\mathbf{v} = \omega \times \mathbf{r}$$

We use a co-ordinate system located at the center of mass which is assumed to be at rest.

1.3.2 The rotational kinetic energy is a quadratic funcion of angular velocity:

$$K = \frac{1}{2}\omega^T A \omega$$

A is a symmetrix matrix called the moment of inertia. We can derive a formula for it in terms of the density of the body

$$K = \frac{1}{2}\int \rho(\mathrm{r})\,[\omega \times \mathbf{r}]^2\, d^3\mathbf{r}$$

$$= \frac{1}{2}\int \rho(\mathrm{r})\left[\omega^2\mathbf{r}^2 - (\omega\cdot\mathbf{r})^2\right] d^3\mathbf{r}$$

$$= \frac{1}{2}\omega_i\omega_j \int \rho(\mathbf{r})\left[r^2\delta_{ij} - r_i r_j\right] d^3\mathbf{r}$$

so that

$$A_{ij} = \int \rho(\mathbf{r})\left[r^2\delta_{ij} - r_i r_j\right] d^3\mathbf{r}.$$

Since kinetic energy is always ≥ 0, the matrix A is positive. That is, its eigenvalues are positive.

1.3.3 The eigenvalues A_1, A_2, A_3 of the moment of inertia are called the principal moments; the eigenvectors can be chosen to point along the axes of an co-ordinate system, called the body centered frame.

1.3.4 In the body centered frame the angular momentum has a simpler formula: $\mathbf{L} = (A_1\omega_1, A_2\omega_2, A_3\omega_3)$

An important point is that this frame rotates with the body and therefore may not be an inertial co-ordinate system. In particular, even if there are no torques acting on the body, the angular momentum defined in this frame may not be conserved: it differs from the angular momentum defined in an inertial frame by a time dependent rotation. But, a rotation leaves the length of a vector unchanged.

1.3.5 The magnitude L of angular momentum is conserved

$$L^2 = L_1^2 + L_2^2 + L_3^2. \tag{1}$$

1.3.6 The rotational kinetic energy of a body is, in the same frame ,

$$K = \frac{L_1^2}{2A_1} + \frac{L_2^2}{2A_2} + \frac{L_3^2}{2A_3} \tag{2}$$

This is similar to the way that the translational kinetic energy is $\frac{\mathbf{p}^2}{2m}$.

1.3.7 The rotational kinetic energy is conserved as well.

1.4 Thus the angular momentum vecor moves along the intersection of a sphere with an ellipsoid.

Imagine a space in which the axes measure the components of angular momentum along the principal directions L_1, L_2, L_3. As **L** is not conserved, the angular momentum vector moves around with time in space. The sphere is the surface of constant magnitude for angular momentum, so the tip of **L** lies on this sphere. The condition $K =$ constant picks out an ellipsoid whose principal radii are $\sqrt{2KA_1}, \sqrt{2KA_2}, \sqrt{2KA_3}$. Thus as time evolves, the angular momentum vector must move along the curve defined by the intersection of these two surfaces.

1.4.1 The simplest case is a body with spherical symmetry: all the eigenvalues are equal. In this case L is a constant.

In this case any right-handed co-ordinate system can be chosen as the principal axes. In fact we do not need complete spherical symmetry to have equal moments of inertia. Any body with the symmetries of a Platonic solid will have equal principal moments of inertia. For example, a cube or a football (soccer ball). In this case, there is no difference between the body centered and inertial co-ordinate systems. $\mathbf{L} = \mathbf{R} =$ constant.

1.4.2 The next simplest case is a body with cylindrical symmetry when two of the eigenvalues are equal.

In this case the body-centered angular momentum around the axis of symmetry (say L_3) is still conserved. Since the sum of the squares of the other two must be fixed, the angular momentum precesses around this axis.

1.4.3 For a cylinder of radius R and height H, $A_1 = A_2 = m\left[\frac{1}{4}R^2 + \frac{1}{3}H^2\right]$, $A_3 = \frac{1}{2}mR^2$.

For cylinder of small height, like a coin $A_1 = A_2 < A_3$.

1.4.4 The most complicated case is when no pair of eigenvalues are equal.

The angular momentum moves along a beautiful but complicated curve called an "elliptic curve". This is because it be parametrized by elliptic functions. Their study is a deep and still evolving branch of algebraic geometry. As usual, in solving algebraic equations, it is useful to continue the equation into complex values of the indeterminates L_1, L_2, L_3. Then we get a complex curve, which is also two dimensional as a real manifold. But we have a complete list of all two dimensional manifolds. From the boundedness of energy and angular momentum we can argue that it is compact. Since time evolution requires the angular momentum to always change with time (except for a few degenerate values of K, L) this manifold must admit a non-zero vector field everywhere. Just on these grounds we can identify what the curve is:

1.4.5 The complex curve defined by the pair of equations (21) is generically a torus.

The physical values are then a real section (one real dimensional submanifold) of this curve. Depending on how you cut it, we can get a circle or two disconnected circles.

2 Euler Equations

We still have not determined the time dependence of the angular momentum vector. The above argument based on conservation laws tells us the shape of the curve followed by the angular momentum vector: but not the rate at which it moves along it, which is not a constant.

Recall that the components of angular momentum satisfy the Poisson brackets of the rotation Lie algebra

$$\{L_1, L_2\} = L_3, \quad \{L_2, L_3\} = L_1, \quad \{L_3, L_1\} = L_2,$$

This along with the formula for energy (2) gives the time derivatives

$$\frac{dL_i}{dt} = \{H, L_i\}$$

These give

2.1 The Euler equations of a rigid body are

$$\frac{dL_1}{dt} = a_1 L_2 L_3, \quad \frac{dL_2}{dt} = a_2 L_3 L_1, \quad \frac{dL_3}{dt} = a_3 L_1 L_2,$$

Here

$$a_1 = \frac{1}{A_2} - \frac{1}{A_3}, \quad a_2 = \frac{1}{A_3} - \frac{1}{A_1}, \quad a_3 = \frac{1}{A_1} - \frac{1}{A_2}$$

etc.

If all the eigenvalues of A are equal, $a_1 = a_2 = a_3 = 0$ and the angular momentum is a constant as expected. If $A_1 = A_2$then L_3 is conserved. In this case

$$\frac{dL_1}{dt} = a_1 L_3 L_2, \quad \frac{dL_2}{dt} = -a_1 L_3 L_1$$

can be solved in terms of trigonometric functions.

2.2 The solution in the general case involves the Jacobi elliptic functions.

2.2.1 These are a set of three functions depending on a paramater k (modulus) and a variable u defined by the differential equatons

$$\frac{d\mathrm{cn}(u,k)}{du} = -\mathrm{sn}(u,k)\mathrm{dn}(u,k), \ d\mathrm{sn}(u,k) = \mathrm{cn}(u,k)\mathrm{dn}(u,k), \ \frac{d\mathrm{dn}(u,k)}{du} = k^2 \mathrm{cn}(u,k)\mathrm{sn}(u,k)$$

To complete the definition, the initial values at $u = 0$ are

$$\mathrm{cn}(0,k) = 1, \ \mathrm{sn}(0,k) = 0, \ \mathrm{dn}(0,k) = 1$$

2.2.2 They satisfy the conditions

$$\mathrm{cn}^2 + \mathrm{sn}^2 = 1, \quad k^2 \mathrm{sn}^2 + \mathrm{dn}^2 = 1.$$

They are well studied and tabulated. Also, if $k = 0$ they reduce to the trigonometric functions.

Exercise 1. Solve the Euler equations in terms of the Jacobi functions using the ansatz

$$L_1 = C_1 \mathrm{cn}(\omega t, k)$$

etc. with appropriate choices of constants C_1, ω, k etc. Determine how C_1, ω, k etc depend on the energy, total angular momentum and the moments of inertia.