Lecture 6

1. Ideal Incompressible Fluids

Euler not only figured out the equations of a rigid body but also those of a fluid. An ideal fluid is a model of fluid motion in which we ignore the effects of friction: in the real world fluids lose energy when different layers have different velocities or when the fluid molecules rub against a boundary. In many cases though the effect is small. Also if the velocity of fluid flow is small compared to the speed of sound of the fluid material, we can treat the density as time independent. The velocity of a fluid is given by a vector field: a vector at any point.

1.1. **A vector field can be thought of as a first order partial differential operator.**

$$u(f) = \mathbf{u} \cdot \nabla f$$

Note that

$$u(f_1 f_2) = f_1 u(f_2) + u(f_1) f_2 \tag{1.1}$$

This is just Leibnitz rule of derivations. Conversely,

1.2. **Any linear operator satisfying (1.1) for all smooth functions is a vector field.** All such operators are of the form $u = \mathbf{u}\cdot\nabla$ for some vector field: (1.1) is the condition that u be a first order derivative operator. So we will from now on say that

1.3. **A vector field is a linear operator satisfying the Leibnitz rule.** The product of two vector fields (thought of as operators) is a second order derivative: no longer a vector field. But,

1.4. **The commutator of two vector fields is also a vector field.** Explicitly, we can see that the second derivatives cancel out in the commutator to give

$$[u, v]f = u(v(f)) - v(u(f))$$

$$[\mathbf{u}, \mathbf{v}] = \mathbf{u} \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \mathbf{u}$$

$$[u, v]^i = u^j \partial_j v^i - v^j \partial_j u^i.$$

All different ways of thinking about the commutator of vector fields.

1.5. **The set of all vector fields is a Lie algebra under the operation of commutator.** Not surprising if we know that vector fields are first order partial differential operators. The only question is whether the commutator of first order operators is still first order.

1.6. **An incompressible vector field satisfies.**

$$\nabla \cdot \mathbf{u} = 0$$

If the density is variable the conservation of mass is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [\rho \mathbf{u}] = 0$$

It says that the net amount of mass flowing into a small region is equal to the increase of mass inside it . If the density is constant we get the above condition. Since it is constant we can use units such that $\rho = 1$.

1.7. **The commutator of incompressible vector fields is incompressible.** Thus the set of incompressible vector fields form a Lie algebra.

The condition of constant density is much weaker than that of a rigid body. Still, this Lie algebra of incompressible vector fields is analogous to the rotations of a rigid body. We can introduce a basis and write the commutation relations in terms of them. But we will need an infinite number of basis elements: there are many more incompressible vector fields than just infinitesimal rotations.

1.8. **In the absence of external forces the total energy is the kinetic energy.**

$$K = \frac{1}{2} \int \mathbf{u}^2 d^3\mathbf{x}.$$

1.9. **The equations of motion of an ideal incompressible fluid follow from the commutator and the kinetic energy.** These are the famous Euler equations. The derivation outlined above is a bit above the level of this course. I refer to you to another set of lectures I gave:

http://arxiv.org/abs/0906.0184.

2. The Quantum Rigid Body

2.1. **There are many molecules whose rotational spectrum is given by rigid body mechanics.** In the case of molecules, the vibrational frequencies are usually much higher than the rotational frequencies. So to a good approximation we can treat some of them as rigid bodies in calculating the rotational spectrum. Of course, no object is strictly rigid. If we apply enough stresses any object can be deformed. If the angular momentum is too large, the centrigual forces will deform the body.

2.2. **The simplest case of a molecule with the symmetries of a Platonic solid is easy to solve.** There are some molecules that have a high degree of symmetry under discrete rotations. A famous example is the "Bucky ball" a molecule made of sixty carbon atoms arranged symmetrically. In such cases the three eigenvalues of moment inertia coincide.

$$H = \frac{L^2}{2A}.$$

The spectrum is

$$E_l = \frac{j(j+1)}{2A}$$

each state being $2l+1$-fold degenerate: the energy does not depend on$m = -j, \cdots j$.

2.3. **The next simplest case has two prinicpal moments equal.**

$$H = \frac{L_1^2 + L_2^2}{2A_1} + \frac{L_3^2}{2A_3}.$$

In this case we can rewrite

$$H = \frac{L_1^2 + L_2^2 + L_3^2}{2A_1} + \left[\frac{1}{2A_3} - \frac{1}{2A_1}\right] L_3^2.$$

The energy eigenvalues are

$$E_{lm} = \frac{j(j+1)}{2A_1} + \left[\frac{1}{2A_3} - \frac{1}{2A_1}\right] m^2, \quad m = -j - j+1, \cdots j-1, j.$$

Note that m and $-m$ have the same energy, so the degeneracy is not completely lifted.

2.3.1. *If $A_1 = A_2 > A_3$ the ground state has $m = 0$ if j is integer and $m = \pm\frac{1}{2}$ if j is half integer.* Then, $A_3^{-1} > A_1^{-1}$ and we want to minimize $|m|$ to get the lowest energy. Thus, for integer angular momentum and $A_1 = A_2 > A_3$ the ground state is unique. These bodies are shaped like a long rod or cigar. The angular momentum will, in the ground state, point perpendicular to the axis of symmetry.

2.3.2. *If $A_1 = A_2 < A_3$ the ground state has $m = \pm j$.* Such a body will put all of its angular momentum to point along the axis of symmetry to minimize the energy.These are shaped like a pancake or frisbee. The energy is the same for either direction of rotation: the actual ground state will be a superposition of the two. Exactly which superposition will depend on finer details: small corrections to energy we are ignoring for now.

2.4. **If there are no symmetries, the principal moments of inertia are unequal and the spectrum is more complicated.** Neverthless we can determine it for small angular momentum by "brute force" diagonalization of the hamiltonian. For large angular momenta the semi-classical approximation ought to be close. In between is a puzzle.

2.4.1. *The energy eigenvalues for total angular momentum j are determined by the equation.*

$$P_l(z_1, z_2, z_3) = 0$$

$$z_a = \frac{1}{2A_a} - \frac{E}{j(j+1)},\ a = 1, 2, 3.$$

$$P(z_1, z_2, z_3) = \det\left[z_1 L_1^2 + z_2 L_2^2 + z_3 L_3^2\right]$$

The characteristic equation of the hamitonian is

$$\det[H - E] = \det[H - \frac{E}{j(j+1)} L^2]$$

and then we split

$$L^2 = L_1^2 + L_2^2 + L_3^2.$$

Thus we have a polynomial of order $2j+1$ in three variables. It is convenient to continue the values of z_a into complex values, as often in solving polynomial equations. Note that this polynomial is symmetric under the permutation of z_1, z_2, z_3: this can be undone by a rotation which is an equivalence transformation that leaves the spectrum unchanged. More interestingly, we have the homogenity

$$P(\lambda z_1, \lambda z_2, \lambda z_3) = \lambda^{2j+1} P(z_1, z_2 z_3).$$

Thus one of the variables (doesn't matter which one) can be removed by scaling.

2.4.2. *The spectrum of the general rigid body of angular momentum j is determined by a complex projective curve of degree $2j+1$.* The equation being homogenous of degree $2j+1$, it can be viewed as a curve in two dimensional projective space P^2.

2.4.3. *For small values of $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \cdots$ this polynomial can be split as a product each of degree < 5.*. The significance of degree being less than five is that then we can solve the polynomials "explictly" in terms of radicals. In splitting the polynomials several partial symmetries are useful. For example, L_1^2, L_2^2 only change the value of L_3 two units:

$$L_1^2 + L_2^2 = L_- L_+ + L_3$$

$$L_1^2 - L_2^2 = \frac{L_+^2 + L_-^2}{2}$$

$$L_1^2 = \frac{1}{2}[L_- L_+ + L_3] + \frac{L_+^2 + L_-^2}{4}$$

etc.

Thus (considering only integer j for simplicity) even and odd values of m do not mix with each other. The polynomial splits into a piece that is the determinant of the matrix in each subspace. Further reductions are possible as well. See Landau and Lifshitz.

Exercise 1. What is the largest value of $j$for which you can split this polynomials down to a solvable one?

2.4.4. *I conjecture that the spectral curve of the rigid body $P_j(z_1, z_2, z_3) = 0$ can be solved in parametric form using the elliptic modular form $\Delta(\tau)$ for any integer j..* This is motivated by the fact that the classical solution can be expressed in terms of elliptic functions. More precisely, the action of the rigid body can be expressed in terms of complete elliptic integrals which in turn are related to Δ.