Lecture 8

1. ROTATIONS THROUGH 2π

Recall that the the rotation group is $SO(3)$, the set of orthogonal matrices of positive determinant. Closely related is the group $SU(2)$ of unitary matrices of determinant one. We saw that infinitesimally they are the same: they have isomorphic Lie algebras. But they are not the same for finite rotations. The difference has to do with rotations through 2π.

1.1. **There is a homomorphism** $R : SU(2) \to SO(3)$**.** That is, an element in $SU(2)$ determines a rotation in a way that preserved multiplication laws. The connection is through 2×2 matrices. Recall that there is a one-one correspondence between vectors in three dimensional space and traceless hermitean matrices:

$$\mathbf{a} \leftrightarrow \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix}$$

We will denote vectors in R^3 by boldface in this section; a is the matrix corresponding to $\mathbf{a}$.

$$a = \sigma_1 a_1 + \sigma_2 a_2 + \sigma_3 a_3$$

Conversely

$$a_i = \frac{1}{2}\mathrm{tr}\sigma_i a$$

Any traceless hermitean matrix determines three real numbers which then can be grouped into vector in R^3. The scalar (dot) product of vectors becomes the trace of products of matrices:

$$\mathbf{a}\cdot\mathbf{b} = \frac{1}{2}\mathrm{tr}\begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix}\begin{pmatrix} b_3 & b_1 - ib_2 \\ b_1 + ib_2 & -b_3 \end{pmatrix}$$

Let us denote the quantity on the r.h.s. (the inner product) by $\langle a, b\rangle$. It is clear that the inner product is unchanged under unitary transformations:

$$< gag^\dagger, gbg^\dagger > = < a, b >$$

So unitary transformations on matrices must induce rotations on vectors:

$$gag^\dagger = \sigma\cdot[R(g)\mathbf{a}]$$

But notice that g and $-g$ both determine the same rotation. For example 1and -1 both go over to the identity. More detail follows from considering rotations around some direction (say third axis). Recall that any rotation can around the third axis can be written as

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = e^{\theta S_3}, \quad S_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In particular, a rotation through 2π is the same as the identity. This comes from the unitary matrix

$$g(\theta) = e^{i\frac{\sigma_3}{2}\theta} = \begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix}$$

through the map Rdefined above.

Exercise 1. Prove that $R(g(\theta)) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Because of the factor of $\frac{1}{2}$ in the exponent we see that a rotation through 2π does not correspond to the identity

$$g(2\pi) = -1.$$

The above $g(\theta)$ for $0 \leq \theta \leq 2\pi$ is a closed curve in $SO(3)$ that starts and ends at the identity is an open curve that starts at the identity and ends at -1. But there is a deeper topological fact we cannot prove here

1.2. **$g(\theta)$ for $0 \leq \theta \leq 2\pi$ is a curve that cannot be continuously deformed to a constant.** That is, there is no continous function $\tilde{g}(\theta, s)$ such that $\tilde{g}(\theta, 0) = 1, \tilde{g}(\theta, 1) = g(\theta)$. But if we traverse this same curve twice, by letting θ vary through $0 \leq \theta \leq 4\pi$ we do get one that can be deformed to the identity! We say that the fundamental group (the set of equivalence classes of curves that can be deformed into each other) of $SO(3)$ is Z_2 : any closed curve traversed twice is deformable to a constant. But in $SU(2)$, all closed curves can indeed be deformed to the constant. The non-deformable closed cuves in $SO(3)$ start and end at some g and $-g$ in $SU(2)$. In opological lanuage we say that $SU(2)$ is the universal cover of $SO(3)$.

1.3. **The map $R : SU(2) \to SO(3)$ is a double covering.** This has profound consequences for physics.

1.4. **Representations of $SU(2)$of integer spin are also representations of $SO(3)$.** Once we have a set of three matrices J_1, J_2, J_3 that satisfy the commutation relations of angular momentum, $e^{iJ_a\theta_a}$ will representations of finite rotations. The tricky point here is that for half-integer spins a rotation through 2π is not represented by the identity, but by -1.

1.5. **Half-integer spin representations of $SU(2)$ are not representations of $SO(3)$.** So why is half-integer spin allowed in nature? It was a surprise when first discovered. It is because of a quirk of quantum mechanics:

1.6. **States of a quantum system correspond to rays in complex Hilbert space.** If we multiply a vector in Hilbert space by a scalar, it does not change the state it represents. Since a vector and its negative describe the same state, a rotation through 2π does not have to be represented by the identity: it can also be represented by -1.

2. BOSONS

2.1. **The state of a quantum system must remain unchanged when a pair of identical particles are interchanged.** Identical means they must have the same characteristics such as spin,mass,angular momentum. Although the remains unchanged, the vector representing it in Hilbert space can change by a scalar multiple. When space is three dimensional, it is possible to interchange two particles by a rotation through π around a perpendicular bisector of the line that connects them. If the particles are identical, the physical effect is the same as rotating a single particle around a circle by 2π radians. Thus there is a deep connection between the behavior of a system under rotations and interchange of identical particles.

2.2. **When a pair identical particles of integer spin are interchanged the state vector is unchanged; for half integer spin it changes by a sign.**

2.3. **Particles whose state vectors are symmetric (anti-symmetric) under interchange are called bosons (fermions).** It is a profound fact of nature that spin and statistics are so intimately related. There could have been other possibilities: the state vector could have changed by some other scalar multiple than just a sign, thus producing more general kinds of statistics (anyons). They are allowed in two space dimensions. It is one of the rare situations that get more complicated in lower dimensions.

2.4. **If the space of states of a single boson is V, that of a pair of bosons is $S^2(V)$, the space of symmetric matrices.** In some orthonormal basis, single particle states are given by vectors $\psi = (\psi_1, \cdots \psi_M)$

while two boson states are

$$\psi_{ij} = \psi_{ji}.$$

For fermions we would have anti-symmetric matrices. Suppose $i = 1, \cdots M$: the single boson has some finite number M of states available to it. Then there are $\frac{M(M+1)}{2}$ independent two-boson states. More generally

2.5. **The space of states of n bosons is $S^n(V)$ the space of symmetric tensors.**

$$\psi_{i_1\cdots i_n} = \psi_{i_1\cdots i_a\cdots i_b\cdots i_n}$$

invariant under any interchange. For fermions we would get anti-symmetric tensors that change sign under odd permutations.

2.6. **The total state space of bosons is $S(V) = \bigoplus_{n=0}^{\infty} S^n(V)$.** $S^0(V) = C$ is the vacuum or empty state, represented by a tensor of rank zero: a scalar.

2.7. **We can also think of $S(V)$ as the space of polynomials.**

$$\psi(z) = \sum_{n=0}^{\infty} \psi_{i_1\cdots i_n} z_{i_1}\cdots z_{i_n}$$

The degree of the polynomial is the total number of bosons. In any given state this is a finite number, but we allow it to be as large as needed. Because the components of the complex numbers commute the coefficients are symmetric tensors.

2.7.1. *In the special case* $\dim V = 1$ *there is a correspondence between free bosonic states and the states of a simple harmonic oscillator.*

$$\psi(z) = \sum_{n=0}^{\infty} \psi_n \frac{z^n}{\sqrt{n!}}$$

We can associate

$$\frac{z^n}{\sqrt{n!}} = |n>, \quad n = 0, 1, \cdots$$

If the single boson has energy ω, a system of n free bosons will have energy $n\omega$: exactly the energy of the state $|n\rangle$ of a simple harmonic oscillator. (We add a constant to the hamiltonian so that the lowest energy state has energy zero.)

Recall that for a simple harmonic oscillator

$$H = \omega a^\dagger a, \quad [a, a^\dagger] = 1$$

A representation is given by

$$a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

where

$$|n\rangle, \quad n = 0, 1, \cdots$$

is an orthonormal basis. Indeed,

$$a^\dagger a|n\rangle = n|n\rangle.$$

2.7.2. *The creation operator is just multiplication by z; destruction is differentiation.*

$$a^\dagger = z, \quad a = \frac{\partial}{\partial z}$$

satisfies the relation. Since

$$\frac{\partial}{\partial z} z^n = n z^{n-1}$$

and

$$a|n\rangle = \sqrt{n}|n-1\rangle$$

the correspondence is

$$|n\rangle = \frac{z^n}{\sqrt{n!}}.$$

Thus the number of bosons occupying the state corresponds to the principal quantum number or to the degree of the polynomial.

2.7.3. *The hamiltonian is a differential operator.*

$$H = \omega z \frac{\partial}{\partial z}$$

2.7.4. *The inner product is given by integration with a Gaussian measure.*

$$||\psi(z)||^2 = \int e^{-|z|^2} \psi^*(z)\psi(z)\frac{d^2z}{\pi}$$

Here $\int d^2z$ denotes integration over the whole complex plane.

Exercise 2. Prove that

$$\int e^{-|z|^2} \frac{z^{*m}}{\sqrt{m!}} \frac{z^n}{\sqrt{n!}} \frac{d^2z}{\pi} = \delta_{mn}$$

Now we can consider the more general case where the single particle has many states available to it.

2.8. **The space of polynomials $S(V)$ carries a representation of the canonical commutation relations.**

$$[a_i, a_j^\dagger] = \delta_{ij}, \quad [a_i, a_j] = 0 = [a_i^\dagger, a_j^\dagger]$$

$$a_i^\dagger = z_i, \quad a_i = \frac{\partial}{\partial z^i}$$

$$|n_1, n_2 \cdots\rangle = \frac{z^{n_1}}{\sqrt{n_1!}} \frac{z^{n_2}}{\sqrt{n_2!}} \cdots$$

$$\psi(z) = \sum_{n_i=0}^{\infty} \psi_{n_1 n_2 \cdots} |n_1, n_2 \cdots\rangle$$

$$||\psi(z)||^2 = \int e^{-\sum_i |z|^2} \psi^*(z)\psi(z) \frac{d^2 z_1 d^2 z_2}{\pi}$$

2.9. **A system for free bosons is the same as a harmonic oscillator.** If the energies of the single bosons are ω_i,the energy of a collection of free bosons is just the sum $n_1\omega_1 + n_2\omega_2 + \cdots$ of individual energies.

$$H = \sum_i \omega_i a_i^\dagger a_i.$$

This is called the "coherent state description" of bosons and harmonic oscillators. See *Quantum Optics* by Klauder and Sudarshan for details.

3. Fermions

The description of bosonic states as polynomials is so compelling that Berezin developed its analogue for fermions. But for this he had to invent a new number system, called *Grassmann numbers.* Grassmann is a pioneer in algebra and geomery in the nineteenth century and anticipated much modern mathematics. The reason why the coefficients of polynomials are symmetric is that complex numbers $z_i$commute with each other. To get anti-symmetric states we must think of polynomials in anti-commuting variables.

3.1. **Grassmann variables satisfy the relations.**

$$\zeta_i \zeta_j + \zeta_j \zeta_i = 0, \quad i = 1, \cdots M$$

In particular when $i = j$ we get

$$\zeta_1^2 = \zeta_2^2 = 0$$

etc.

3.1.1. *Suppose there is only one such variable satisfying* $\zeta^2 = 0$. The only polynomial is then

$$\psi(\zeta) = \psi_0 + \psi_1 \zeta$$

The coefficients ψ_0, ψ_1 are complex numbers. On the other hand a state eithe empty or it is occupied by one fermion (exclusion principle). This corresponds exactly to the polynomial: the vacuum is the first term and the one-particle state is the second.

3.2. **The most general polynomial in Grassmann variables is.**

$$\psi(\zeta) = \sum_{n=0}^{M} \psi_{i_1 \cdots i_n} \zeta_{i_1} \cdots \zeta_{i_n}, \quad \psi_{i_1 \cdots i_n} = (-1)^P \psi_{i_{P_1} \cdots i_{P_n}}$$

Here P is a permutation and $(-1)^P$ is one if an even number of indices are switched and -1 for an odd number.

Note that there cannot be more than M indices: we cannot satisfy the anti-symmetry condition.

Exercise 3. Show that he number of independent components of a completely anti-symmetric tensor of order n is $\begin{pmatrix} M \\ n \end{pmatrix} = \frac{M(M-1)\cdots(M-n+1)}{n!}$. Explain the meaning of the identity $\sum_{n=0}^{M} \begin{pmatrix} M \\ n \end{pmatrix} = 2^M$ in terms of Pauli's exclusion principle.