Lecture 9

1. COMPACT LIE ALGEBRAS

1.1. **An inner product on a real vector space is a positive bilinear function $\langle .,. \rangle : V \times V \to R$.** That is

$$< u, v >=< v, u >$$

$$< u, u >\geq 0, \quad < u, u >= 0 \implies u = 0.$$

$$< \alpha_1 u_1 + \alpha_2 u_2, v >= \alpha_1 < u_1, v > +\alpha_2 < u_2, v >$$

1.2. **A Lie algebra is said to be compact if it has an invariant inner product.** Invariant means that

$$< [u, v], w > + < v, [u, w] >= 0.$$

This is the infinitesimal version of the condition that the inner product be invariant under the action of a group.

Example 1. Recall that $o(3)$ is the Lie algebra of vectors in R^3 under the cross product. Then the dot product is an invariant inner product

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} + \mathbf{v} \cdot (\mathbf{u} \times \mathbf{w}) = 0.$$

Thus $o(3)$is an invariant inner product.

Example 2. More generally, $o(n)$ (the space of antisymmetric real matrices under commutator) is a compact Lie algebra. There is an invariant inner product

$$< a, b >= -\mathrm{tr}\ ab$$

The negative sign makes it positive $-\mathrm{tr}a^2 = \mathrm{tr}a^T a \geq 0$. Also

$$\mathrm{tr}[a, b]c + \mathrm{tr}b[a, c] = \mathrm{tr}[a, bc] = 0.$$

Exercise 3. $u(n)$ is a compact Lie algebra as well. Recall that it is the space of anti-hermitean matrices. What is the inner product?

Example 4. $su(n)$ is a compact Lie algebra.

2. IDEALS

2.1. **The Direct Sum of two vector space V_1 and V_2 is the set of ordered pairs with pairwise addition and multiplication.**

$$(u_1,u_2)+(v_1,v_2)=(u_1+v_1,u_2+v_2)$$

$$\alpha(u_1,u_2)=(\alpha u_1,\alpha u_2)$$

2.2. **The Direct Sum of two Lie algebras has commutators defined through each component as well.**

$$[(u_1,u_2),(v_1,v_2)]=([u_1,v_1],[u_2,v_2])$$

2.3. **The Direct Sum of Lie algebras correspond to the Direct Product of Lie Groups.**

$$(g_1,g_2)(h_1,h_2)=(g_1h_1,g_2h_2)$$

etc. Roughly speaking a Lie group is the exponential of its Lie algebra. That is why one operation is called sum and the other the product.

2.4. **A subalgebra of a Lie algebra is a subspace that is closed under the Lie bracket.**

2.5. **If $H\subseteq G$ is a subspace such that $u\in H, v\in G \implies [u,v]\in H$ then H is an ideal of G.** Note that an ideal is a special kind of subalgebra.

Exercise 5. $su(n)$ (traceless matrices) is a Lie subalgebra of $u(n)$. Is it an ideal?

2.6. **A Lie Algebra is said to be simple if it has no proper ideal.** Proper means that the ideal is not either the trivial subspace or the whole space: these cases are uninteresting, of course. Simple here does not mean easy to understand. It means indvisible. In hindsight a name like "Prime Lie Algebra" would have been better than "Simple Lie Algebra".

2.7. **A Lie Algebra is semi-simple if its proper ideals are abelian.** Thus $u(n)$ is not semi-simple: the multiples of the identity are an abelian ideal.

2.8. **Any compact Lie algebra is the semi-direct sum of a semi-simple Lie algebra with an abelian Lie algebra.** If the Lie algebra G has a representation on a vector space V, the semidirect sum is the Lie algebra with bracket

$$[(a,v),(a',v')]=([a,a'],av'-a'v)$$

2.9. **Any compact semi-simple Lie algebra is the direct sum of simple Lie algebras.** The orthogonal complement of an ideal in such a Lie algebra is also an ideal: so we can split it as a sum. We can iterate this until we run out of ideals.

2.10. $su(n), o(n)$ **are simple Lie algebras. There is a similar sequence of algebras called** $sp(n)$ **made of matrices with quaternionic entries.**

2.11. **This is almost a complete list of compact simple algebras: there is just a finite number of exceptional algebras in addition to these.** These have names like G_2, F_2, E_6, E_7, E_8 and are related to octonions (a number system whose multiplication not associative, but whose commutator still satisfies the Jacobi identity.) Despite many valiant attempts these have not been found to be very useful in physics.

2.12. **It looks like physics prefers ordinariness rather than exceptionality.**

2.13. **Compact Simple Lie algebras and** $U(1)$ **are the building blocks of fundamental theories of physics.** The structure theory of compact simple Lie groups due to Cartan is one of the gems of mathematics. It is beautiful looked at from any angle. These groups show up in physics over and over: first in nuclear physics($SU(2)$: Heisenberg and Wigner) then particle physics ($su(3)$: Gell-Mann, Okubo) then in unified theories ($SU(2) \times U(1)$ Glashow, Salam, Weinberg), attempts at Grand Unification ($SU(4)$:Pati,Salam, $SU(5)$ Georgi, Glashow) and string theory (E_8 Gross,Witten). This theory is the toolkit of model builders: the phenemenologists who try to fit observations into a unified theory. See the book by Georgi for much more on this.

Example 6. $u(n)$ is not simple: the subalgebra of matrices that are proportional to the identity is an ideal.

2.14. **Any compact Lie algebra is the direct sum of simple and abelian Lie algebras.**

Exercise 7. $u(n) = R \oplus su(n)$.

Thus we can get a complete list of all compact Lie algebras if we can find all the compact simple algebras. Mathematicians (like botanists or librarians) love to organize things into lists and classify them. Cartan gave the complete list of compact Lie algebras, and in particular discovered the exceptional Lie algebras.

2.15. **The rank of a Lie algebra is the largest number of linearly independent commuting elements.**

Exercise 8. The rank of $u(n)$ is n. The rank of $su(n)$ is $n-1$.

Exercise 9. The rank of $o(2k)$ is k and that of $o(2k+1)$ is k as well.

2.16. **A Cartan subalgebra of a Lie algebra is an abelian subalgebra of maximal dimension.** The rank of a Lie algebra is the dimension of a Cartan subalgebra.

2.17. **It is convenient to use an orthonormal basis of the invariant inner product.**

$$[e_a, e_b] = f_{abc} e_c$$

2.18. **The invariance of the inner prodct implies that the structure constants f_{abc} are completely anti-symmetric.** That

$$f_{abc} = -f_{bac}$$

follows from the anti-symmetry of the Lie bracket. The additional anti-symmetry

$$f_{abc} + f_{acb} = 0$$

follows from invariance of the inner product. (Prove this!)

3. $SU(3)$

3.0.1. *The group $SU(3)$ is of much interest in particle physics.*

3.1. $K^{\pm}$ **are pseudo-scalar , isospin $\frac{1}{2}$ particles of mass** 494 **Mev that only decay by weak interactions.** They were called "strange particles" when they were discovered. What was strange about them is that they were unusually long lived ($10^{-8}s$): suggesting that they carry a quantum number that is approximately conserved. This number was called "strangeness" (Gell-Mann). They are each other's anti-particles. K^+ was assigned strangenesss $S = +1$and therefore K^- would have $S = -1$. Unlike $\pi^{\pm}$ the $K^{\pm}$ have form an isospin $\frac{1}{2}$ doublet.

3.2. $K^0, \bar{K^0}$ **is another pair of pseudo-scalar , isospin $\frac{1}{2}$ particles of mass** 498 **MeV that are also stable under strong interactions.** K^0 has $I_3 = -\frac{1}{2}, S = 1$ and $\bar{K^0}$has $I_3 = \frac{1}{2}, S = -1$ .The charges of all the K-mesons can be fit by changing the formula for electric charge (Gell-Mann-Nihijima)

$$Q = I_3 + \frac{B+S}{2}$$

3.3. **The new quantum number is counts the net number of a new kind quark, the strange quark.** By a twist of fate, the strange quark has $S = -1$ and the strange anti-quark hs $S = +1$. It has baryon number $\frac{1}{3}$ like the u and d quarks. From the above formula we see that its electric charge is $-\frac{1}{3}$. That is the same charge as the d quark. Thus we have the constituents of the Kaons:

$$K^+ = \bar{s}u,\ K^- = \bar{u}s,\ K^0 = \bar{s}d\ \bar{K^0} = \bar{d}s$$

3.4. **There is also a neutral pseudoscalar meson that has strangeness zero and isospin zero.**

$$\eta^0 = \bar{s}s$$

with a mass ≈ 548MeV. It decays mostly into 2γ which can be thought of as the strange quark and anti-quark annihilating each other. A more accurate description of the η^0 includes mixing with $\bar{u}u$ and $\bar{d}d$. More on mixing later.

3.5. **The s quark is heavier than the $u$and d quarks.** which explains why particles that contain it as a few hundred MeV heavier than corresponding particles made from u and $d$quarks alone.For example, $m_{K^+} - m_{\pi^+} \approx 350$ MeV.Recall that the d quark is slightly heavier (by a few MeV) than the u quark to explain the neutron-proton mass difference. For strong interactions, the three quarks behave the same way. If we also ignore their mass differences, the isospin symmetry is enlarged to a symmetry that rotates three quarks into each other. Since these transformations can involve complex matrices, the symmetry must involve 3×3 complex matrices. One natural choice is to generalize the $SU(2)$ of isospin to $SU(3)$. This is not the only possibility: there several rank two Lie groups (with two commuting quantum numbers such as I_3 and S) to choose from. But $SU(3)$ is what worked.

3.6. **The 8 pseudo-scalar mesons form a representation of $SU(3)$.** Analogous to the way the pions form a three dimensional representation of $SU(2)$.

3.7. **A basis for $su(2)$ is provided by the Pauli matrices.**

3.7.1. *More precisely any traceless hermitean 2×2 matrix can be written as.*

$$A = a_1 \frac{\sigma_1}{2} + a_2 \frac{\sigma_2}{2} + a_3 \frac{\sigma_3}{2}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The coefficients a_i are real. The factor of i is needed to make the term anti-hermitean. the factor of $\frac{1}{2}$ is a convention, which assures that

$$\operatorname{tr} A^{\dagger}A = a_1^2 + a_2^2 + a_3^2.$$

Thus the Pauli matrices provide an orthonormal basis of the invariant inner product in $su(2)$.

3.7.2. *We can think of the up and down quarks as eigenstates of* σ_3*with eigenvalues* ± 1.

3.7.3. *As more quarks were added, the approximate symmetry of isospin came to be enlarged to larger groups. The up, down and strange quarks correspond to* $su(3)$.

3.8. **The number of linearly independent elements of $u(n)$ is n^2.** A hermitean mtrix has n^2 independent components: there are n real entries along the diagonal and $\frac{n(n-1)}{2}$ complex numbers above the diagonal. The entries below the diagonal are not independent because they are just complex conjugates of the ones above, so the total is $n + 2\frac{n(n-1)}{2} = n^2$. Since an anti-hermitean matrix is simply i times a hermitean one, its number of independent components is also n^2.This is called the dimension of $u(n)$.

3.8.1. *The dimension of* $su(n)$ *is* $n^2 - 1$. The condition of being traceless imposes one condition among the diagonal entries, so the number of independent components of $su(n)$ is $n^2 - 1$.

3.8.2. *The dimension of* $su(3)$ *is* 8. *Its rank is* 2.

3.9. **The Gell-Mann matrices provide a basis for $su(3)$.**

$$A = a_1\frac{\lambda_1}{2} + a_2\frac{\lambda_2}{2} + a_3\frac{\lambda_3}{2} + a_4\frac{\lambda_4}{2} + a_5\frac{\lambda_5}{2} + a_6\frac{\lambda_6}{2} + a_7\frac{\lambda_7}{2} + a_8\frac{\lambda_8}{2}$$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

These are normalized such that

$$\mathrm{tr}\, \lambda_\alpha \lambda_\beta = 2\delta_{\alpha\beta}.$$

3.9.1. *λ_3 and λ_8 are diagonal. They span a Cartan subalgebra of $su(3)$.*

Exercise 10. Derive the commutation relations of $su(3)$in the Gell-Mann basis. That is, write the commutators $\left[\lambda_\alpha, \lambda_\beta\right] = i f_{\alpha\beta\gamma}\lambda_\gamma$ as linear combinations ofthe Gell-Mann matrices. Identify subsets of generators that are transformed among each other by the basis of the Cartan subalgebra..