PHY 407 QUANTUM MECHANICS Fall 05
Problem set 1 Due Sep 15 2005

1. Let V be the set of all complex valued functions of a real variable θ, that are periodic with period 2π. That is

$$u(\theta + 2\pi) = u(\theta), \text{for all } u \in V. \tag{1}$$

(i) Show that this V is a complex vector space.

(ii) What is the dimension of this vector space?. Prove your answer.

(iii) Define $< u|v >= \int_0^{2\pi} u^*(\theta)v(\theta)d\theta$. Show that this satisfies all the conditions for an inner product.

(iv) Find a pair of functions that are orthogonal with respect to this inner product.

(v) Let $b_n = e^{in\theta}$, where n is any integer. Calculate the inner product $< b_m|b_n >$ for any pair of such functions. (*Hint*:Pay special attention to the case $m = n$.)

(vi) Find an orthonormal basis for V.

2. Let V be the space of Problem 1. Consider the operator $\frac{1}{i}\frac{d}{d\theta}$ on V .

(i) Is this operator hermitean?

(ii) What are its eigenvectors and eigenvalues?

3. Consider the following two matrices:

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad N = \begin{pmatrix} 5 & 3a \\ 6 & b \end{pmatrix}. \tag{2}$$

(i) For what values of a and b are these simultaneously diagonalizable?

(ii) What are the simultaneous eigenvectors and eigenvalues of M and N in this case?

PHY 407 QUANTUM MECHANICS Fall 05
Problem set 2 Due Sep 22 2005

4. Consider the Poisson bracket on the phase space of a system with one degree of freedom:

$$\{f, g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial q}\frac{\partial f}{\partial p}. \tag{3}$$

Show that this satisfies the following properties:

(i) $\{f, g\} = -\{g, f\}$

(ii) $\{f, gh\} = \{f, g\}h + g\{f, h\}$

(iii) $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$

Can you think of an operation among matrices that happens to have these three properties?.

5. Let $\vec{L} = \vec{r} \times \vec{p}$ be the angular momentum of a particle in three dimensional space. Calculate the Poisson brackets of the components of $\vec{L}$. (i.e., $\{L_1, L_2\}$ etc.)

6. Consider a system with a two dimensional space of states and hamiltonian

$$H = \begin{pmatrix} a & b \\ b^* & d \end{pmatrix} \tag{4}$$

where a, d are real numbers.

(i) Find the eigenstates and eigenvalues of the hamiltonian.

(ii) Assuming that the initial state of the system is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, what is the probability of finding it in the state

$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ after a time t?

(iii) Give an example of a physical system for which the two state description is a good approximation.

PHY 407 QUANTUM MECHANICS Fall 05
Problem set 3 Due Sep 29 2005

7. Recall that classically, angular momentum is the cross product of the position vector with the momentum vector.

(i) Use the correspondence principle to obtain operators describing the three components of angular momentum.

(ii) Derive an inequality for the uncertainty in the measurement of two components of angular momentum.

8. (i) Express the kinetic energy operator of a particle moving in a plane in polar co-ordinates.

(ii) Find the energy levels of a particle of mass m confined to move in a circle of radius R.

(iii) Solve the Schrödinger equation of a free particle moving in the plane in polar co-ordinates.

9. Find the state on which the product of the uncertainties in the measurement of position and momentum of a particle with one degree of freedom is a minimum. Prove that any other state will have a larger value for this product.

PHY 407 QUANTUM MECHANICS Fall 05
Problem set 4 Due Oct 6 2005

10. Consider the hartmonic oscillator hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2. \tag{5}$$

Find the expectation value $< \psi_0|x^4|\psi_0 >$ in the ground state wavefunction ψ_0.

Hint Express x in terms of creation and annihilation operators.

11. Determine the ground state eigenfunction of the hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \alpha|x| \tag{6}$$

Hint Use the fact that the ground state will be symmetric $\psi(x) = \psi(-x)$ and hence $\psi'(0) = 0$ .Find the asymptotic behavior as $|x| \to \infty$; then use the power series method to derive a recursion relation, and solve it.

Do one of the following two problems; they are harder

12. Let $aa^\dagger - a^\dagger a = 1$ as usual. What is the spectrum of the hamiltonian

$$H = \omega a^\dagger a + z a^{\dagger 2} + z^* a^2 \tag{7}$$

where z is some complex number?

13. Suppose we modify the canonical commutation relations to

$$aa^\dagger - q a^\dagger a = 1 \tag{8}$$

where q is a real number. What is the spectrum of the analogue of the harmonic oscillator hamiltonian, $H = a^\dagger a$?

PHY 407 QUANTUM MECHANICS Fall 05
Problem set 5 Due Oct 13 2005

14. Consider a crude model for a 'molecule', in which an 'electron' is attracted towards two 'nuclei' located a distance R apart. Assume that the 'electron' only moves along the line passing through the two 'nuclei' and that the attractive potential between the electron and the nuclei is of zero range.

(i) Find the hamiltonian of this system.

(ii) Find the ground state energy and the ground state wavefunction.

Hint The transcendental equation you will get for the energy cannot be solved explicitly. You can get a plot of the solution numerically. Or, you can get a solution in parametric form (i.e., both the energy and the coupling constant are expressed as functions of the same parameter.)

(iii) What is the force between the 'nuclei' due to the exchange of the 'electron'? If the 'nuclei' repel each other according to Coulomb's law $V_{\text{Coul}} = \frac{\alpha}{R}$, what is the length of the chemical bond in the molecule?

15. In same system as above, find the first excited state and its energy. What is the force between the 'nuclei' when the 'electron' is in this state? Can a molecule form in this state?

16. Use the variational method to find an approximation to the ground state energy of a particle of mass

m moving in one dimension, under the influence of the potential $V(x) = -\alpha e^{-a|x|}$. Recover the answer for the delta-function potential in the limit $a \to \infty, \alpha \to \infty$ keeping $\frac{a}{\alpha}$ fixed.

PHY 407 QUANTUM MECHANICS Fall 05
Problem set 6 Due Oct 20 2005

17. Find the best variational estimate you can of the ground state energy of the anharmonic oscillator

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2 + gx^3 + \lambda x^4 \tag{9}$$

18. The hamiltonian below has two classical ground states:

$$H = \frac{1}{2}p^2 + \lambda(x^2 - a^2)^2 \tag{10}$$

(i) Find the variational estimate of the quantum ground state energy based on a Gaussian ansatz concentrated on one of the minima of the potential.

(ii) Then find the estimate based on a linear combination of two Gaussians, each concentrated on one of the minima.

(iii) Which has lower value of the energy? What does this suggest about the expectation value $< x >$ of the position in the true ground state?

19. Recall that the eigenvalue problem of the hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \alpha|x|. \tag{11}$$

can be solved in terms of Airy functions.

(i) Estimate the ground state energy using a Gaussian ansatz; compare with the exact value.

(ii) Estimate also the first excited state energy by using an appropriate ansatz; compare again with the exact value.

PHY 407 QUANTUM MECHANICS Fall 05
Problem set 7 Due Oct 27 2005

20.Find the energy levels within the WKB approximation for $H = \frac{p^2}{2m} + V(x)$ for the following potentials:

(i) $V(x) = \lambda|x|^k$, for $-\infty < x < \infty$, $\lambda, k > 0$

(ii) $V(x) = -\frac{\alpha}{x}$, for $\alpha, x > 0$

(iii) $V(x) = -\frac{\alpha}{x} + \frac{l(l+1)}{x^2}$, for $l, \alpha, x > 0$. Compare with those situations where an exact answer is available.

21. Consider the hamiltonian $H = \frac{p^2}{2m} + V(x)$ for a potential $V(x)$ that goes to zero faster than $\frac{1}{|x|}$ as $|x| \to \infty$ and is negative and bounded everywhere.("Shallow attractive potential"). Use the WKB approximation to find a formula for the number of bound states of energy less than E. In particular, estimate the total number bound states (i.e., the number of states with energy < 0).

22. Consider the hamiltonian $H = \frac{p^2}{2m} + V(x)$ with potential $V(x) = -\frac{\alpha}{|x|} + kx$, which models the motion of an electron in the presence of a nucleus and a constant external field. Use the variational principle to show that there is no ground state (i.e., that there are states of energy can be as small as you wish). Nevertheless, show that the lifetime of an electron bound to the nucleus can be quite large, by estimating the probability of escape in the semi-classical approximation.

PHY 407 QUANTUM MECHANICS Fall 05
Problem set 8 Due Nov 17 2005

23. What is the probability that a particle **in the ground state** of a simple harmonic oscillator will be found at a position that is classically forbidden?

24. A particle of mass m moves in a potential

$$V(r) = \frac{A}{r^2} - \frac{B}{r} \tag{12}$$

where $A, B \geq 0$. Find the energy levels of this system by first deriving an equation for the radial wave function for each value of total angular monetum l, and then finding the condition for a square integrable radial eigenfunction. Compare the case $A = 0$ with that of a hydrogenic atom.

25. A rigid body has hamiltonian

$$H = \frac{L_x^2}{2I_1} + \frac{L_y^2}{2I_2} + \frac{L_z^2}{2I_3} \tag{13}$$

where L_x, L_y, L_z satisfy the commutation relations of angular momentum and $0 < I_1 \leq I_2 \leq I_3$ are the principal moments of inertia. Consider states with a given total angular momentum $L^2 = \hbar^2 l(l+1), l \geq 0$.

(i) If $I_1 < I_2 = I_3$ find the eigenvalues; what are the states of least and greatest energy?

(ii) Same as above when $I_1 = I_2 < I_3$.

(iii) Find all the eigenvalues for the asymmetrical top $I_1 < I_2 < I_3$ and $l = 2$. You need to first find the hamiltonian as a 5×5 matrix using the matrix elements of the angular momentum operators. Its secular equation will decompose into three linear equations and a quadratic equation.

PHY 407 QUANTUM MECHANICS Fall 05
Problem set 8 Due Dec 1 2005

26. Two particles of spin $\frac{1}{2}$ interact through the hamiltonian

$$H = a\sigma_1\sigma_1' + b\sigma_2\sigma_2' + c\sigma_3\sigma_3' \tag{14}$$

where σ, σ' denote the Pauli spin matrices acting on the states of the first and second particles respectively. Find the eigenstates and eigenvalues of the hamiltonian. Here, $0 < a < b < c$.

27. Consider the 2×2 hamiltonian

$$H = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix} + g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(i) Find exact eigenstates and eigenvalues as a function of g.

(ii) Find the eigenvalues to the second order in g using perturbation theory. Compare with the expansion of the exact answer to second order.

(iii) If the eigenvalues are expanded in a power series of g what will be the radius of convergence of this series? (Pay attention to the special case $\epsilon_1 = \epsilon_2$.)

28. Find the ground state energy of the anharmonic oscillaor

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + gx^4 \tag{15}$$

upto **second order** in perturbation theory.