PHY 411 Advanced Classical Mechanics (Chaos)

Problem set 1 Fall 2006

Due Sep 28th 2006

Please put your solutions in my mail box by the end of the day on Wednesday.

1. (i) Let $(x(t), y(t))$ be a curve in the plane. Define its action to be

$$S = \frac{1}{2}\int_0^T [\dot{x}^2 + \dot{y}^2]dt$$

i.e., the action of a free particle moving on the plane. What are the functions $x(t), y(t)$ that minimize the action?

(ii) Now suppose we change the formula for the action to

$$S = \frac{1}{2}\int_0^T \left[\frac{\dot{x}^2 + \dot{y}^2}{y^2}\right] dt, \quad y > 0$$

This is the action of a particle moving on a hyperboloid, written in Poincare co-ordinates. Again find the equation for the curve that minimizes S. Solve these equations and plot some so that you can identify their geometrical meaning.

2. Define the Poisson bracket of a pair of functions f and g on the plane to be

$$\{f, g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p}.$$

(i) Show that the time derivative of any function is given by

$$\frac{df}{dt} = \{H, f\}$$

(ii) Prove the identities

$$\{f, g\} = -\{g, f\}, \quad \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0, , \quad \{p, q\} = 1.$$

3. In quantum mechanics observables are represented by operators on a complex Hilbert space. What is the analogue of the Poisson brackets in quantum mechanics? Find the analogue of each identity above and prove it in quantum mechanics.

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Problem set 2 Fall 2006

Due Oct 17th 2006

4. Find the action-angle variables of the Kepler problem with Hamiltonian

$$H = \frac{1}{2m}\left(p_r^2 + \frac{1}{r^2}p_\theta^2\right) - \frac{GMm}{r}$$

by solving the Hamilton-Jacobi equation using separation of variables. what is the hamiltonian as a function of the action variables?

5. Suppose T is a matrix all of whose entries are positive.

(i) What can you say about the eigenvalues and eigenvectors of T? (e.g., Are they always real? Is there an eigenvector with positive components? Does it have the largest eigenvalue?)

(ii) Given an initial vector u_0, define $u_1 = Tu_0, u = Tu_1$ etc. What can you say about the limit of this sequence u_n as $n \to \infty$?

(iii) For the matrix $T = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ find the eigenvectors and eigenvalues; find the limit of the above sequence for an arbitrary starting vector u_0 by decomposing it in terms of eigenvectors.

6. Consider the function $f : [0,1] \to [0,1]$ given by $f(x) = 2x \bmod 1$; i.e., double x and take its fractional part. Define also the space of sequences $\Sigma = \{s_0 s_1 \cdots | s_j = 0, 1\}$ and the shift map $\sigma : \Sigma \to \Sigma$ by $[\sigma(s)]_j = s_{j+1}$.

(i) Find a map $\phi : [0,1] \to \Sigma$ with respect to which f is conjugate to the shift map: $\phi \circ f = \sigma \circ \phi$.

(ii) How many solutions are there to the equation $f^m(x) = x$, where $f^m(x)$ is m th of iteration of f ?

(iii) Given a map $f : X \to X$ on any space X , define the *dynamical zeta function* to be $\zeta_f(z) = \exp\left[\sum_{m=1}^{\infty} \frac{z^m}{m} N_m(f)\right]$ where $N_m(f)$ is the number of fixed points of the m th iteration of f . It is a sort of generating function for the sequence of numbers $N_m(f)$. Find the dynamical zeta function for the shift map σ defined above.

Solutions

4. The Hamilton-Jacobi equation is

$$\frac{1}{2m}\left(\left[\frac{\partial S}{\partial r}\right]^2 + \frac{1}{r^2}\left[\frac{\partial S}{\partial \theta}\right]^2\right) - \frac{GMm}{r} = E$$

Putting in the ansatz for the separation of variables

$$S(r,\theta) = R(r) + \Theta(\theta)$$

gives

$$\frac{1}{2m}\left(\left[\frac{\partial R}{\partial r}\right]^2 + \frac{1}{r^2}\left[\frac{\partial \Theta}{\partial \theta}\right]^2\right) - \frac{GMm}{r} = E$$

Differentiating w.r.t. θ we see that $\frac{\partial \Theta}{\partial \theta}$ is a constant. If we choose the boundary condition $\Theta(0) = 0$,

$$\Theta(\theta) = p_\theta \theta$$

The action variable correspodnint to this co-ordinate is given by the total change in Θ over one orbit; i.e., as θ varies from 0 to 2π:

$$J_\theta = 2\pi p_\theta.$$

For radial motion,

$$\frac{1}{2m}\left(\left[\frac{\partial R}{\partial r}\right]^2 + \frac{p_\theta^2}{r^2}\right) - \frac{GMm}{r} = E$$

or

$$R(r) = \pm \int^r \left(2mE + \frac{GMm^2}{r} - \frac{p_\theta^2}{r^2}\right)^{\frac{1}{2}} dr$$

The indefinite integral can be evaluate in terms of elliptic functions. For $E < 0$, the radial action variable is the total change in R over one orbit; i.e., twice the change as r varies from its smallest value r_1 to its largest value r_2:

$$J_r = 2\int_{r_1}^{r_2} \left(2mE + \frac{GMm^2}{r} - \frac{p_\theta^2}{r^2}\right)^{\frac{1}{2}} dr$$

Here $r_1 < r_2$ are the roots of the quantity under the square root; i.e., of the quadratic equation

$$2mEr^2 + GMm^2r - p_\theta^2 = 0.$$

We can evaluate this integral by contour integration for the case of bound orbits for which $E < 0$.

Put

$$A = 2m|E|, \quad B = GMm^2, \quad C = p_\theta^2.$$

If we analytically continue the function

$$f(r) = \left(-A + \frac{B}{r} - \frac{C}{r^2}\right)^{\frac{1}{2}}$$

to the complex r-plane it has a square root branch cut along the interval $[r_1, r_2]$ on the real axis. Let C be a contour that surrounds this interval in the anti-clockwise direction. Then the lhs is just $\int_C f(r)dr$. The function also has a simple pole at $r = 0$ and a pole at infinity. Thus we can use the residue theorem to get

$$\int_C f(r)dr = 2\pi i \left[a_0 + a_\infty\right]$$

where a_0, a_∞ are the residues of the poles.

The pole at the origin evidently has residue $a_0 = \sqrt{(-C)}$. To understand the behavior at infinity we make the change of variable $u = \frac{1}{r}$. Putting $f(r)dr = g(u)du$,and expanding arounf $u = 0$,

$$g(u) = -\frac{1}{u^2}\left(-A + Bu - Cu^2\right)^{\frac{1}{2}} \approx -\frac{\sqrt{(-A)}}{u^2}\left[1 - \frac{B}{2A}u + \cdots\right].$$

we get the residue

$$a_\infty = \frac{-B}{2\sqrt{(-A)}}.$$

Thus

$$2\int_{r_1}^{r_2}\left(-A + \frac{B}{r} - \frac{C}{r^2}\right)^{\frac{1}{2}} dr = -2\pi\left[\sqrt{C} + \frac{B}{2\sqrt{A}}\right]$$

Thus

$$I_r = -2\pi \left[p_\theta + \frac{GMm^2}{2\sqrt(2m|E|)} \right]$$

or

$$I_r + I_\theta = -\pi \frac{GMm^2}{\sqrt(2m)} \frac{1}{|E|}.$$

Thus, the hamiltonian or energy is

$$H(I_r, I_\theta) = -\frac{\pi^2 G^2 M^2 m^3}{2(I_r + I_\theta)^2}$$

5(i)

• Given an $n \times n$ matrix T,we say that ψ is an eigenvector with eigenvalue λ if $T\psi = \lambda\psi$ with $\psi =\neq 0$. An eigenvalue must be a root of the characteristic polynomial

$$P_T(z) = \det[T - z].$$

Conversely to each root there is at least one eigenvector. Even if T is real, the eigenvalues may not be real: they will occur as complex conjugate pairs though in that case. the eigenvectors corresponding to unequal eigenvalues are linearly independent. If there are n distinct eigenvalues, the eigenvectors form a basis. However they are not necesarily orthogonal to each other. The proofs can be found in many books on linear algebra.

•Suppose T is a matrix with non-degenerate roots for characteristic polynomial. Let S be a matrix whose kth column is the eigenvector ψ_k. Due to linear independence the ψ_k, S is invertible. Also, the eigenvalue equation gives

$$T = S\Lambda S^{-1}, \quad \Lambda = \text{diag}\ \{\lambda_1, \cdots \lambda_n\}$$

Now it is clear that

$$\text{tr}\ T = \sum_k \lambda_k, \text{tr}\ T^2 = \sum_k \lambda_k^2, \cdots \text{tr}\ T^m = \sum_k \lambda^m \cdots$$

and

$$\det T = \prod_k \lambda_k.$$

• Even a matrix T with positive entries does not necessarily have real eigenvalues. An example is the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 9 & 4 & 1 \\ 25 & 16 & 9 \end{pmatrix}$$

which by numerical solution can be seen to have the spectrum

$$13.7566, 0.121676 + 2.15348i, 0.121676 - 2.15348i.$$

• However, two by two matrices with positive entries, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ always have real eigenvalues. By explicit solution of the eigenvalue equation we get $\frac{a+d}{2} \pm \frac{1}{2}\sqrt{[(a-d)^2 + 4bc]}$.

• A matrix with positive entries will always have an eigenvector with positive entries. This can be proved using the Brouwer fixed point theorem along the lines mentioned in class. Define the simplex

$$\Delta_{n-1} = \{(u_1, \cdots u_n) | u_1 \geq 0, \cdots u_n \geq 0, \sum_i u_i = 1\}.$$

A matrix with positive entries defines a map $\tilde{T} : \Delta_n \mapsto \Delta_n$

$$[\tilde{T}u]_i = \frac{[Tu]_i}{\sum_k [Tu]_k}.$$

Being homeomorphic to a closed ball, anycontinuous map of $\Delta_n \to \Delta_n$ must have a fixed point according to the Brower fixed point theorem. This fixed point will satisfy,

$$Tu = \lambda u, \lambda = \sum_k [Tu]_k.$$

Hence this is an eigenvector with positive entries.

• Define the norm of a real vector as $|u| = \sup_i |u_i|$: i.e., the magnitude of the largest component of u. Define also T by

$$|T| = \sup_{|u|=1} |Tu|.$$

Then $|T|$ is the square root of the largest eigenvalue of $T^\dagger T$. *It is important that the norm of the vector used here is* not *the same as the length.* If T has positive entries, this maximum is achieved by a vector with positive components.

(ii) Let us assume for simplicity that the roots of the characteristic polynomial of T are distinct. Then the eigenvectors form a basis and we can expand $u = \sum_k c_k \psi_k$. Then $Tu = \sum_k \lambda_k c_k \psi_k$ and $T^m u = \sum_k \lambda^m c_k \psi_k$. As m becomes large only the eigenvalue λ_1 with the largest magnitude will survive. For,

$$T^m u = \lambda_1^m \left[c_1 \psi_1 + \sum_{k \neq 1} \left(\frac{\lambda_k}{\lambda_1} \right)^m \psi_k \right]$$

As $m \to \infty$,

$$T^m u \to \lambda_1^m c_1 \psi_1.$$

(iii) The eigenvalues of the given matrix are by explicit solution, $1 \pm \sqrt{2}$. The eigenvector of the eigenvalue $1 \pm \sqrt{2}$ is

$$\begin{pmatrix} \pm\sqrt{2} \\ 1 \end{pmatrix}.$$

We can determine the coefficients by expanding

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = c_1 \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$$

Thus

$$c_1 = \frac{1}{2\sqrt{2}} u_1 + \frac{1}{2} u_2, \quad c_2 = -\frac{1}{2\sqrt{2}} u_1 + \frac{1}{2} u_2.$$

Thus

$$T^m u \to (1 + \sqrt{2})^m [-\frac{1}{2\sqrt{2}} u_1 + \frac{1}{2} u_2] \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$$

as $m \to \infty$. 6. Any number in the interval $[0, 1]$ can be expanded in its binary expansion:

$$x = \sum_{k=1}^{\infty} \frac{s_k}{2^k}, \quad s_k = 0, 1.$$

$x = 0.s_1 s_2 s_2 \cdots$. Multiplication by two just shifts the sequence by one step $0.s_1 s_2 s_2 \cdots \mapsto s_1.s_2 s_3 \cdots$ taking it modulo one means we just drop anything to the left of the binary point. Thus the map $f(x) = 2x \bmod 1$ is conjugate to the shift map

$$\phi : s_0.s_1 s_2 s_2 \cdots \mapsto 0.s_2 s_3 \cdots$$

i.e., $s_k \mapsto s_{k+1}$. If we iterate the map m times,

$$\phi^m : s_k \mapsto s_{k+m}.$$

Thus a fixed point of this iterate would be repeat after m steps:

$$.s_0 s_1 \cdots s_m s_1 s_2 \cdots$$

Since each s_k can take two values $0, 1$ there are $N_m(f) = 2^m$ fixed points. The zeta function is

$$\begin{aligned} \zeta_f(z) &= \exp\left[\sum_{m=1}^{\infty} \frac{z^m}{m} 2^m\right] \\ &= \exp(-\log[1 - 2z]) \\ &= \frac{1}{1 - 2z}. \end{aligned} \tag{1}$$

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Problem set 3 Fall 2006

Due Oct 31st 2006

7. For a relativistic particle moving in a potential $V(x)$ the relation between spatial momentum p and energy p_0 is $[p_0 - V]^2 - c^2\mathbf{p}^2 = m^2c^4$. Suppose V is the Coulomb potential: $V(r) = -\frac{Z\alpha}{r}$. Write down the relativistic Hamilton-Jacobi equation for motion in a plane. Express the energy p_0 in terms of the action variables of the motion. Apply the Bohr-Sommerfeld quantization rule (that the action must be an integer multiple of Plank's constant) to get the spectrum of a hydrogenic atom. Show that this relativistic effect partially lifts the degeneracy of the levels of hydrogen.

8. (i) Define the Riemann zeta function by $\zeta(s) = \sum_1^\infty \frac{1}{n^s}$. Show that this is an analytic function on the half plane Re$s > 1$. Prove the Euler product formula $\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$ where the product ranges over all prime numbers.

(ii) Define the Mobius function $\mu(n)$ by $\frac{1}{\zeta(s)} = \sum_{n=1}^\infty \frac{\mu(n)}{n^s}$. Obtain a formula for $\mu(n)$ in terms of the decomposition of n as a product of primes. Series of the type $\sum_{n=1}^\infty \frac{a_n}{n^s}$ are called Dirichlet series.

9. Let $f : X \to X$ be a continuous map on some space. We say that a positive number n is the period of x if it is the *smallest* number such that $f^n(x) = x$. Let ν_n be the number of orbits of period n and N_n the number of solutions to the equation $f^n(x) = x$.

(i) Clearly any multiple m of n will also satisfy $f^m(x) = x$, so there is a relation between N_n and ν_n. Show that $\sum_{d|n} d\nu_d = N_n$ where the sum ranges over the divisors of n.

(ii) Define $\zeta_f(z) = \exp\left[\sum_{m=1}^\infty \frac{z^m}{m} N_m\right]$ as before. Show that $\zeta_f(s) = \prod_{n=1}^\infty \frac{1}{(1-z^n)^{\nu_n}}$.

(iii) Express ν_n in terms of N_n using the Mobius function $\mu(n)$ defined above.

10. Consider a graph of two vertices with incidence matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$; i.e., every pair of vertices is an edge. For paths on this graph, find the number N_m of closed paths of length m. Relate it to A^m. Find the zeta function $\exp[\sum_{m=1} \frac{z^m}{m} N_m]$ and express it terms of the matrix A.

PHY 411 Advanced Classical Mechanics (Chaos)

Problem set 4 Fall 2006

Due Nov 28th 2006

11.This problem concerns the kernel of the heat equation on the circle; i.e.,

$$\frac{\partial}{\partial t}h_t(x) = \frac{\partial^2}{\partial x^2}h_t(x)$$

subject to the periodic boundary consition $h(x+1) = h(x)$ and the initial condition $\lim_{t\to 0^+} h_t(x) = \delta(x)$.

(i) Find a solution using the Fourier series $h_t(x) = \sum_{n=-\infty}^{\infty} a_n(t)e^{2\pi inx}$.

(ii) Another method is to start with the solution $\tilde{h}_t(x)$ on the real line (i.e., boundary conditions $\lim_{|x|\to\infty} h_t(x) = 0$) and showing that $h_t(x) = \sum_{n=-\infty}^{\infty} \tilde{h}_t(x+n)$.

(iii) Let $g(t) = h_t(0)$. Use the two methods above to establish a relation between large and small times for $g(t)$.

(iv) Recall the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$. Use the above identity to derive a relation between $\zeta(s)$ and $\zeta(1-s)$.

12. Consider the iteration of the map of the unit interval to itself: $f(x) = 2x \bmod 1$. Suppose there is a small random error η in each step of this iteration. Assume that the error is a Gaussian random variable of zero mean,variance σ^2, and that the error in each iteration is independent of the others. Derive the equation for the evolution of the probability distribution of the point on the interval. Solve this equation using Fourier series and find the distribution for large time.

13. A particle moves on the line under the influence of a viscous damping force , a random force $\eta(t)$ and a potential $V(x)$:

$$m\frac{du}{dt} = -\gamma u + \eta(t).$$

Assuming that the random force is rapidly varying, of zero mean and that the variance of $B(\Delta t) = \int_t^{t+\Delta t} \eta(t)dt$ is $\sigma\Delta t$, derive a differential equation for the evolution of the probability density in velocity space. Solve this equation and find the asymptotic distribution for large time.