Lecture 1

1. RELATIVITY

1.1. **It is an astonishing physical fact that the speed of light is the same for all observers.** This is not true of other waves. For example, the speed of sound measured by someone on standing on Earth is different from that measured by someone in an airplane. This is because sound is the vibration of molecules of air while light is the oscillation of electric and magnetic fields: there is no medium (ether) needed for its propagation. If an observer is moving with velocity v_1 relative to air, the speed of sound measured by him/her would be

$$v = v_1 + c_s$$

where c_s is the speed of sound measured by a static observer.

1.2. **The law of addition of velocities has to be modified to take account of this fact.** The usual law (Galeleo) for addition of velocities (considering only one component for simplicty)

$$u + v$$

would lead to velocities greater than or less than c , the speed of light. The correct law is

$$\frac{u+v}{1+\frac{uv}{c^2}}.$$

1.3. **Rapidity is a more convenient variable than velocity in relativistic mechanics.** If either u or v is of magntitude c, the sum is also of magnitude c. If we make the change of variables (considering only one component for simplicity)

$$v = c \tanh \theta$$

this formula becomes simple addition:

$$\tanh[\theta_1 + \theta_2] = \frac{\tanh \theta_1 + \tanh \theta_2}{1 + \tanh \theta_1 \tanh \theta_2}.$$

The variable θ is called *rapidity.* Although velocities cannot exceed c,rapidity can be as big as you want. As $\theta \to \infty$, $v \to c$. This also suggests that a boost (change of velocity) is some kind of rotation, through an imaginary angle. Remember that $\tan i\theta = i \tanh \theta$.

1.4. **The wavefront of light has the same shape for all observers.** Imagine that you turn on and off quickly a light bulb at $t = 0$,at the position $\mathbf{x} = 0$. Light propagates along cone

$$c^2t^2 - x_1^2 - x_2^2 - x_3^2 = 0,$$

1.5. **The laws of physics must be the same for all observers.** This is the principle of relativity. It is not a discovery of Einstein: what is new is really how he reconciled this principle with the fact that the velocity of light is the same for all observers. Einstein realized that Newton's laws of mechanics need to be modified to fit with the new law for the addition of velocities. Minkowski realized that the theory of relativity can be understood geometrically: it says that the square of the distance between two events (points in space-time) is

$$c^2(t-t')^2 - (x_1 - x_1')^2 - (x_2 - x_2')^2 - (x_3 - x_3')^2$$

Lorentz transformations (changes of velocities) are like rotations in the $x - t$ plane. Because of the relative sign difference between the time and space components, these rotations are through an imaginary angle; this angle is rapidity. More generally

1.6. **The Minkowski inner product of four-vectors is.**

$$u \cdot v = u_0v_0 - u_1v_1 - u_2v_2 - u_3v_3$$

It is useful to write this as

$$u \cdot v = u^T \eta v$$

where

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

is called the *Minkowski metric.*

Remark 1. Be warned that all physicists except particle physicists, use the opposite convention, with three positive and one negative sign. Even some particle physicists use the opposite convention (e.g., S. Weinberg).

The points of space time are four vectors with components ct, x_1, x_2, x_3.

1.7. **Rapidity can be thought of as an angle in the $x\ t$ plane.** Again consider just one direction for space. Supose $x\ t$ and $x'\ t'$ are space and time as measured by two observers. Since they must agree that the velocity of light is the same, the shape of the wave front must be the same as well:

$$c^2t^2 - x^2 = c^2t'^2 - x'^2$$

This means that

$$x' = \cosh\theta x - \sinh\theta ct$$

$$ct' = \cosh\theta ct - \sinh\theta x$$

for some real number θ. Similar to the way a rotation in the plane leaves the square of the distance $x^2 + y^2$ unchanged. We can identify this angle with rapidity. Along $x' = 0$ (the position of one of the observers) the relation between $x$and t is

$$\frac{x}{t} = c\tanh\theta.$$

In other words $v = c\tanh\theta$. The above is an example of a Lorentz transformation. More generally,

1.8. **A Lorentz transformation is a 4×4 matrix that leaves the Minkowski distance unchanged.** Thus a Lorentz transformations is much like a rotation, except that the matrices must satisfy the condition

$$\Lambda^T\eta\Lambda = \eta.$$

1.9. **The products and inverses of Lorentz transformations are also Lorentz transformations.** This means that the set of Lorentz transformations forms a *group*. It is denoted by $O(1,3)$: orthogonal matrices with respect to a metric $\eta$with 1 positive sign and three negative signs.

1.10. **This is analogous to the group of rotations in Euclidean space.** What are the linear transformations in R^3 that leave the length of vectors unchanged? The square of the length of a vector can be thought of as $a^T a = a_1^2 + a_2^2 + a_3^2$. Thus

$$a' = Ra, \quad a'^T a' a = a^T a$$

is satisfied if

$$R^T R = 1.$$

Such matrices are said to be **Orthogonal.** The product and inverse of orthogonal matrices is also orthogonal: the set of orthogonal matrices is a group. It is denoted by O(3). Rotations are orthogonal matrices which also have determinant one.

1.11. **O(3) is contained as a special case of O(1,3).** A matrix that does not mix space and time

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix}$$

is a Lorentz transformation if R is orthogonal.

1.11.1. *We can split Lorentz transformations into four types using the signs of* $\det\Lambda$ *and* Λ_{00}.. If $\Lambda_{00} < 0$ the transformation includes a time reversal. If $\Lambda_{00} > 0$ and $\det\Lambda < 0$, it includes a space reversal (Parity). The subgroup with

$$\det L > 0, \quad \Lambda_{00} > 0$$

is called the set of *proper Lorentz transformations* $SO(1,3)$. Although at first all four types of Lorentz transformations appeared to be at he same footing, it turned out neither Parity nor time reversal is an exact symmetry of nature.

1.11.2. *Only the proper Lorentz transformtions are exact symmetries of nature.* Parity is violated by the weak interactions: essentially the fact that neutrinos are left handed. Time reversal is violated by the phase in the mass matrix (Kobayashi-Mazkawa matrix) of quarks.

1.11.3. *Infinitesimal Lorentz transformations form a six dimensional Lie algebra.* This means that the commutator of two infinitesimal Lorentz transformations is also an infinitesimal Lorentz transformation.

If $\Lambda = 1 + \lambda$ for some small matrix λ, the Lorentz condition becomes

$$\lambda^T \eta + \eta\lambda = 0$$

This means that $\eta\lambda$ is anti-symmetric. A four by four antis-symmetric matrix has six independent components. Three of them represent infinitesimal rotations. The remaining three are boosts (changes of velocity) in the three co-ordinate directions. Any infinitesimal Lorentz transformation can be written as the sum of the six independent matrices:

$$L_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad L_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad L_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

(1.1)

$$L_{01} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad L_{02} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad L_{03} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

They satisfy the commutation relations

$$[L_{12}, L_{23}] = -L_{13}$$

$$[L_{12}, L_{01}] = L_{02}$$

$$[L_{01}, L_{02}] = -L_{12}$$

and their cyclic permutations. These commutation relations define the Lorentz Lie algebra. They can also be written as

$$[L_{\mu\nu}, L_{\rho\sigma}] = \eta_{\nu\rho} L_{\mu\sigma} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho} + \eta_{\mu\sigma} L_{\nu\rho}$$

1.11.4. *In addition to Lorentz transformations, translations are also symmetries. Together they form a ten dimensional algebra of symmetries, the Poincare' algebra.*

$$[L_{\mu\nu}, L_{\rho\sigma}] = \eta_{\nu\rho} L_{\mu\sigma} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho} + \eta_{\mu\sigma} L_{\nu\rho}$$

$$[L_{\mu\nu}, P_\rho] = \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu$$

$$[P_\mu, P_\nu] = 0. \tag{1.2}$$

1.12. **Invariance under translations leads to the conservation of energy and momentum.**