1. **The Wave Equation**

1.1. The amplitude of a small wave propagating with speed $c$ satisfies.

$$\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = 0$$

1.1.1. *Plane waves are solutions*

$$\phi(x) = e^{i(\omega t - k \cdot x)}, \quad \frac{\omega^2}{c^2} - k^2 = 0$$

1.2. In Lorentz invariant form the wave equation is.

$$\eta^{\mu\nu} \partial_\mu \partial_\nu \phi = 0$$

Remember that all wave equations are invariant under Lorentz transformations; even sound. But there is something special about light: the speed is the same for all observers. Relativity is much more than invariance under Lorentz transformations.

1.3. The wave equation follows from a variational principle.

$$S = \frac{1}{2} \int \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi dx$$

1.3.1. *Here $dx$ stands for the volume measure of space-time $dx^0 dx^1 dx^2 dx^3$.* Just like in mechanics, except that the function depends on several variables.

$$\delta S = \int \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \delta \phi dx = \int \partial_\nu [\eta^{\mu\nu} \partial_\mu \phi] \delta \phi dx - \int [\eta^{\mu\nu} \partial_\mu \partial_\nu \phi] \delta \phi dx$$

By using Gauss’ theorem (that the integral of the divergence of a vector field is equal to flux through the boundary) the first term depends only on the boundary. We assume that the variation $\delta \phi = 0$ at the boundary; this is analogous to requiring that the variation should vanish at the initial and final points in mechanics. Thus the condition that $\delta S = 0$ is the wave equation.
1.4. Under nonlinear change of co-ordinates the volume measure changes by the Jacobian determinant. Recall that the Jacobi matrix appears in the infinitesimal change of co-ordinates

\[ dx' = \frac{\partial x'}{\partial x} \, dx \equiv J_{\mu}^\nu \, dx^\mu \]

and that the change in volume measure involves the Jacobian

\[ dx' \equiv dx^0 \, dx^1 \, dx^2 \, dx^3 = \det J \, dx \]

1.5. The determinant of the metric tensor transforms with the square of the Jacobian.

\[ g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}, \quad g' = J^{-1} g J^{-1T} \]

\[ \det[g'] = [\det J]^2 \det g \]

1.5.1. The metric tensor of space-time has negative determinant. There are three negative eigenvalues (space) and one positive eigenvalue (time).

1.6. The combination \( \sqrt{-\det g} \, dx \) is invariant under co-ordinate transformations. The determinants cancel out. If the metric is positive we would not put in the negative sign.

1.6.1. In spherical polar co-ordinates,

\[ ds^2 = dr^2 + r^2 \left[ d\theta^2 + \sin^2 \theta d\phi^2 \right] \]

\[ \sqrt{g} \, dx = r^2 \sin \theta \, dr \, d\theta \, d\phi \]

1.7. The generally covariant version of the action for the wave equation is.

\[ S = \frac{1}{2} \int g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \, \sqrt{-\det g} \, dx \]

The combination \( g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \) is a scalar: is invariant under co-ordinate changes. The last part \( \sqrt{-\det g} \, dx \) is invariant as well.

1.8. The generally covariant version of the wave equation is.

\[ \partial_\mu \left[ \sqrt{-\det g} g^{\mu\nu} \partial_\nu \phi \right] = 0 \]

As above

\[ \delta S = \int g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \, \sqrt{-\det g} \, dx = \int \partial_\nu \left[ g^{\mu\nu} \partial_\mu \phi \, \sqrt{-\det g} \right] dx - \int \partial_\nu \left[ g^{\mu\nu} \partial_\mu \phi \, \sqrt{-\det g} \right] \phi dx \]

Again the first term is zero because \( \delta \phi = 0 \) on the boundary.

1.8.1. But we could have obtained a generally covariant wave equation by replacing partial derivatives by covariant derivatives.

\[ g^{\mu\nu} D_\mu D_\nu \phi = 0 \]
1.8.2. This happens to be equivalent to the one above.

\[ \frac{1}{\sqrt{-\det g}} \partial_\mu \left[ \sqrt{-\det g} g^{\mu\nu} \partial_\nu \phi \right] = g^{\mu\nu} D_\mu D_\nu \phi \]

**Proof.** First, recall that the covariant derivative and partial derivative are the same for a scalar. Thus

\[ g^{\mu\nu} D_\mu D_\nu \phi = g^{\mu\nu} \partial_\mu \partial_\nu \phi - g^{\mu\nu} \Gamma^\rho_{\mu\nu} \partial_\rho \phi \]

Now,

\[ g^{\mu\nu} \Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\mu\nu} g^{\rho\sigma} \left[ \partial_\lambda g_{\sigma\mu} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\mu\nu} \right] \]

\[ = g^{\mu\nu} \partial_\sigma g_{\mu\nu} g^{\rho\sigma} - \frac{1}{2} g^{\rho\sigma} [g^{\mu\nu} \partial_\sigma g_{\mu\nu}] \]

Next, recall that the infinitesimal variation of the inverse of a matrix is related to its own variation by

\[ d[A^{-1}] = -A^{-1}[dA]A^{-1} \]

Thus

\[ g^{\mu\nu} \partial_\sigma g_{\mu\nu} g^{\rho\sigma} = -\partial_\sigma g^{\mu\nu} \]

and

\[ g^{\mu\nu} \partial_\sigma g_{\mu\nu} g^{\rho\sigma} = -\partial_\sigma g^{\mu\nu} \]

On the other hand the variation of the determinant of a matrix can be calculated using

\[ \log \det A = \text{tr} \log A \]

\[ \frac{\partial \mu \det A}{\det A} = \text{tr} A^{-1} \partial_\mu A \]

Thus

\[ g^{\mu\nu} \partial_\sigma g_{\mu\nu} = \partial_\mu \log [-\det g] \]

(Switching the sign only shifts the log by a constant.) and

\[ \frac{1}{2} [g^{\mu\nu} \partial_\sigma g_{\mu\nu}] = \partial_\mu \log \sqrt{-\det g} = \frac{\partial_\mu \sqrt{-\det g}}{\sqrt{-\det g}}. \]

Pulling all this together

\[ g^{\mu\nu} D_\mu D_\nu \phi = g^{\mu\nu} \partial_\mu \partial_\nu \phi + [\partial_\mu g^{\rho\sigma}] \partial_\rho \phi + \frac{1}{\sqrt{-\det g}} \left[ \partial_\mu \sqrt{-\det g} \right] g^{\mu\nu} \partial_\nu \phi. \]

The r.h.s. is the same as

\[ \frac{1}{\sqrt{-\det g}} \partial_\mu \left[ \sqrt{-\det g} g^{\mu\nu} \partial_\nu \phi \right] \]

expanded out. \( \square \)
1.9. The wave equation in curved space time is.
\[ g^{\mu \nu} D_\mu D_\nu \phi = 0 \]
For calculations the equivalent form \( \frac{1}{\sqrt{-\det g}} \partial_\mu \left[ \sqrt{-\det g} g^{\mu \nu} \partial_\nu \phi \right] = 0 \) is more convenient.

2. Maxwell’s Equations in Curved Space-Time

2.1. Recall that Maxwell equations in Lorentz covariant form are.
\[ \partial_\mu F^{\mu \nu} = j^\nu, \quad F^{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \]

2.2. They follow from the variational principle.
\[ S = \frac{1}{4} \int F^{\mu \nu} F_{\mu \nu} dx + \int j^\nu A_\nu dx \]
First,
\[ \delta S = \int F^{\mu \nu} \partial_\mu \delta A_\nu dx + \int j^\nu \delta A_\nu dx \]
Now integrate by parts the first term.

2.3. This leads to a wave equation with source for the electromagnetic potential.
\[ \partial_\mu \partial^\mu A^\nu - \partial^\nu [\partial_\mu A^\mu] = j^\nu \]
It is common to impose the condition \( \partial_\mu A^\mu = 0 \) (the Lorentz gauge) taking advantage of the gauge invariance \( A_\mu \rightarrow A_\mu + \partial_\mu \Lambda \). Then each component of \( A_\mu \) satisfies the wave equation
\[ \partial_\mu \partial^\mu A^\nu = j^\nu \]

2.4. The generally covariant form of Maxwell’s equations is.
\[ D_\mu F^{\mu \nu} = j^\nu, \quad F^{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \]
Recall that the Christoffel symbols cancel out in the antisymmetric derivative of a covariant vector.

2.5. In terms of potentials.
\[ D_\mu D^\mu A^\nu - D^\nu A_\mu = j^\mu \]
We cannot interchange the derivatives in the second term without introducing some terms involving curvature.

2.6. An equivalent form of the curved space Maxwell’s equations is.
\[ \frac{1}{\sqrt{-\det g}} \partial_\mu \left[ \sqrt{-\det g} g^{\mu \rho} g^{\nu \sigma} F_{\rho \sigma} \right] = j^\nu \]

2.7. This follows from the covariant variational principle.
\[ S = \frac{1}{4} \int F_{\mu \nu} F_{\rho \sigma} g^{\mu \rho} g^{\nu \sigma} \sqrt{-\det g} dx + \int j^\mu A_\mu \sqrt{-\det g} dx \]
2.8. These equations tell us how the gravitational field affects the propagation of light. For example it can tell us how light is diffracted and refracted by a gravitational field. Spectacular phenomena such as gravitational lensing follow from this. More later.