GRAVITATION F10

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Lecture 14

1. Gravitational Waves

1.1. **There are exact solutions to Einstein's equations that describe gravitational waves.** Could it be that gravitational waves are only approximate solutions to Einstein's equations, and not of the complete theory? To be sure that it is a true prediction, we must verify that such exact solutions exist.

1.1.1. *Start with an ansatz that describes the propagation of a wave along a co-ordinate axis.*

$$ds^2 = dt^2 - dx^2 - \delta_{ab}dx^a dx^b + f(t-x, y, z)(dt-dx)^2$$

Here $a = 2, 3$ label the remaining directions of space, transversal to the direction of the wave. Simultaneous translation in t, x is an isometry. This is a sort of plane wave. If we could solve Einstein's equations to determine f we would have found gravitational plane wave solutions. It is convenient to use as co-ordinates

$$u = \frac{t-x}{\sqrt{2}}, v = \frac{t+x}{\sqrt{2}}, x^a$$

$$ds^2 = 2dudv + 2f(u, y, z)du^2 - \delta_{ab}dx^a dx^b$$

Note that the metric tensor is not diagonal in these "light-cone" co-ordinates.

1.1.2. *To calculate the Christoffel symbols we can use the variational principle for the geodesics.* It is useful to understand the geodesic equations in any case. So this way of calculating it gives us some extra insight into the geometry.

$$S = \frac{1}{2}\int \left[2\dot{u}\dot{v} + 2f\dot{u}^2 - \dot{x}^a\dot{x}^b\delta_{ab}\right] d\tau$$

Varying,

$$\ddot{u} = 0, \quad \ddot{x}^a + \partial_a f\dot{u}^2 = 0.$$

$$\frac{d}{d\tau}\left[\dot{v} + 2f\dot{u}\right] - f'\dot{u}^2 = 0 \implies \ddot{v} + f'\dot{u}^2 + 2\partial_a f\dot{u}\dot{x}^a = 0$$

Comparing with the geodesic equation we see that the only non-zero components of the Christofel symbol are

$$\Gamma^v_{uu} = f', \quad \Gamma^a_{uu} = \partial_a f \quad \Gamma^v_{au} = \partial_a f$$

1.1.3. *Now we calculate the curvature.* The only non-zero components are

$$R^a_{buu} = \partial_b \Gamma^a_{uu} = \partial_a \partial_b f$$

and their permutations.

Lowering the index and remembering the sign of the spatial components of the metric

$$R_{auub} = -\partial_a \partial_b f \implies R_{aubu} = \partial_a \partial_b f$$

Thus the Ricci tensor is identically zero except for the component R_{uu} which is

$$R_{uu} = \partial_a \partial_a f.$$

We get a solution to Einstein's equations if f satisfies the Laplace equation in the two transverse variables $x^2 \equiv y, x^3 \equiv z$. The simplest solutions, which are constants or linear functions of y, z are just flat space in disguise: a co-ordinate transformation will reduce them to Minkowski space. (The curvature vanishes for them.)The next simplest are quadratic functions

$$f(u, y, z) = yz f_1(u) + \frac{1}{2}\left[y^2 - z^2\right] f_2(u)$$

This describes a plane wave with two polarization states. The functions $f_1(u), f_2(u)$ describe the shape of the wave front: the wave dependence on $t - x$. They could for example be periodic (continuos wave), or a wave pulse (like a gaussian). We shouldthink of this as the gravitatinal field created by a very distant source, so far away that the sphere of constant distance from it can be approximated by a plane.

1.2. **Gravitational waves have not yet been detected.** The difficulty is two fold, both ultimately because gravity is such a weak force. Only the most massive bodies moving violently can produce gravitational waves of appreciable magnitude. Only the largest and most sensitive detectors have any hope of seeing their effect. The sources are distant (collapsing stars, blackholes orbiting each other) and rare. Several experimental groups are trying to detect these gravitational waves. Some of the cleverest ideas in engineering and physics go into building these detectors. Supplementing these efforts are massive computer simulations that calculate the wave profiles (for example the functions f_1 and f_2) that woud be seen on Earth.

1.3. **Geodesic Equations in a Gravitational Wave.** Now we can go back and get solutions of the geodesic equations. We see that $\dot{u}$ is a constant, say k. It is useful to change variables and think of the transverse components as functions of u instead of τ.(They only differ by a factor of k anyway.) The transverse components satisfy

$$x''^a + \partial_a f = 0$$

$$\begin{pmatrix} y'' \\ z'' \end{pmatrix} + A(u) \begin{pmatrix} y \\ z \end{pmatrix} = 0, \quad A(u) = \begin{pmatrix} f_2 & f_1 \\ f_1 & -f_2 \end{pmatrix}$$

We are familiar with equations of this type from quantum mechanics , solid state physics or optics. (Bloch waves, Floquet's theorem etc.) There is a solution with initial velocity zero and some position $y(u_0) = y_0, z(u_0) = z_0$. Suppose the wave starts at flat space ($f_2(u_1) = 0$) and ends there after a time u_1: (so that $f_2(u_1) = 0$).

Then the final position $y_1 = y(u_1), z_1 = z(u_1)$ is not in general the same as the initial position: it can be computed knowing the profile of $f_{1,2}(u)$ in the interval $[u_0, u_1]$ in terms of a matrix P, a kind of scattering matrix:

$$\begin{pmatrix} y_1 \\ z_1 \end{pmatrix} = P \begin{pmatrix} y_0 \\ z_0 \end{pmatrix}$$

Thus, freely falling particles initially at rest will have a small displacement in their co-ordinates as the wave passes through. In practice the departure of $P(u)$ from the identity is very small: 10^{-18} or so for astronomical sources.

The principle behind the gigantic detectors that have been built is to use two arms of an optical inteferometer to detect this small displacement. The larger the range of y_0, x_0, the larger the displacement: this is why the detectors are several kilometers long. Some day we hope to have satellites orbiting the Earth or the Sun and measure their relative displacement as a way to detect gravitational radiation. Then we can have hundreds of kilometers for y_0, which will allow us to detect fainter waves caused by more distant sources. There has been a great deal of progress in the theoretical prediction (by numerical solution of Einstein's equations) of the wave shapes $f_{1,2}(u)$ for realistic astronomical sources: a pair of blackholes falling into each other, or a star being swallowed by a blackhole.

1.4. **Perhaps the atom interferometers now made possible using Bose condensates provide new tools to detect small displacements.** New ideas are sorely needed in experimental gravity.