GRAVITATION F10

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Lecture 16

1. The Schwarschild Solution

1.1. **Now we can calculate the components of the Ricci tensor of a centrally symmetric metric.** We don't need the full Riemann tensor for now. Of more immediate interest to us are the components of the Ricci tensor. It is best to use the explicit formula

$$R_{\mu\nu} = \frac{\partial \Gamma^{\rho}_{\mu\nu}}{\partial x^{\rho}} - \frac{\partial \Gamma^{\rho}_{\mu\rho}}{\partial x^{\nu}} + \Gamma^{\rho}_{\mu\nu}\Gamma^{\sigma}_{\rho\sigma} - \Gamma^{\sigma}_{\mu\rho}\Gamma^{\rho}_{\nu\sigma}$$

We leave it as an exercise to calculate these components. Setting them to zero will yield the vaccum Einstein equations.

1.1.1.

1.2. **The centrally symmetric vacuum Einstein equations reduce to a system of ordinary differential equations.**

$$e^{-\lambda}\left[\frac{\nu_r}{r} + \frac{1}{r^2}\right] - \frac{1}{r^2} = 0$$

$$e^{-\lambda}\left[\frac{\lambda_r}{r} - \frac{1}{r^2}\right] + \frac{1}{r^2} = 0$$

$$\lambda_t = 0$$

There is one more equation but it is not independent of these. So we see that λ is independent of time. This is the first half of Birkhoff's theorem. But adding the other two equations give

$$\lambda_r + \nu_r = 0$$

so that

$$\lambda + \nu = f(t)$$

for some function of t alone. Now we pull out the fact that we still have a freedom to make the change of variables

$$t \mapsto T(t)$$

This will change the metric coefficients

$$\nu \mapsto \nu + 2\log \dot{T}, \quad \lambda \mapsto \lambda + 2\log \dot{T}$$

So by choosing T appropriately we can set $f(t) \mapsto 0$.

1.2.1. *All time dependence can be transformed away. This is Birkhoff's theorem.* Now we just have one independent variable that only depends on r. The equations is easily solved with the b.c. that $e^{\lambda} \to 1$ as $r \to \infty$.

$$\left(e^{-\lambda}\right)_r + \frac{e^{-\lambda}}{r} = \frac{1}{r} \implies e^{-\lambda} = 1 + \frac{r_0}{r} = e^{\nu}$$

Here r_0 is a constant of integration with the dimensionsof length, called the Schwarschild radius.

1.2.2. *The constant can be fixed by comparison to Newton's theory.* Recall that $g_{00} \approx 1 + 2\phi$ inthe Newtonian approximation and that $\phi = -\frac{km}{r}$ where k in Newton's constant, where m is the total mass of the centrally symmetric body. Comparing, we get

$$r_S = km$$

Putting in factors of c explictly (recall that $\frac{km}{r}$ has dimensions of the square of velocity)

$$r_S = \frac{km}{c^2}.$$

For the Sun, this is about 10 km. For the Earth, 5 cm.

1.3. **Thus we are led to the Schwarschild solution to vaccum Einstein equations.**

$$ds^2 = \left(1 - \frac{r_S}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{r_S}{r}} - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Far from the center this is the metric of a star or mass m. As we get closer the departure from Newtonian gravity becomes larger. Soon we will solve the geodesic equation to get the corrections to Kepler's laws.

1.4. **Something goes wrong as we approach the Schwarschild radius.** The metric tensor is a diagonal matrix one of whose entries diverges as $r \to r_S$. For a long time this was a source of confusion. Upon careful examination we see that in fact the geometry itself is smooth; it is just that the co-ordinate system is breaking down at $r = r_S$. This is similar to the Euclidean metric of the plane in spherical polar co-ordinates at the origin:

$$ds^2 = dr^2 + r^2 d\theta^2$$

There is nothing special at $r = 0$, we see by transforming to Cartesian co-ordinates.

2. Painleve Co-ordinates

2.1. **We can make a change of co-ordinates that removes the divergent term at $r = r_S$.** We change just the time co-ordinate, a way that can depend on radius $t \mapsto f(T, r)$:

$$dt = f_T dT + f_r dr$$

$$ds^2 = \left(1 - \frac{r_S}{r}\right)[f_T dT + f_r dr]^2 - \frac{dr^2}{1 - \frac{r_S}{r}} - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

We can choose $f$such that the potential divergent term cancels:

$$\left(1 - \frac{r_S}{r}\right) f_r^2 dr^2 - \frac{dr^2}{1 - \frac{r_S}{r}} = -dr^2$$

Then we would have

$$ds^2 = \left(1 - \frac{r_S}{r}\right)\left[f_T^2 dT^2 + 2 f_r f_T dr dT\right] - dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

We can also choose $f_T = 1$ so that there are minimal changes to the other terms in the metric.

$$ds^2 = \left(1 - \frac{r_S}{r}\right)\left[dT^2 + 2 f_r dr\right] - dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

$$\left(1 - \frac{r_S}{r}\right) f_r^2 = -1 + \frac{1}{1 - \frac{r_S}{r}} \implies \left(1 - \frac{r_S}{r}\right) f_r = \pm\sqrt{\frac{r_S}{r}}$$

It turns out that the negative sign corresponds to the blackhole.We get

2.2. **The Schwarschild geometry in Painleve co-ordinate is smooth at $r = r_S$.**

$$ds^2 = \left(1 - \frac{r_S}{r}\right) dT^2 - 2\sqrt{\frac{r_S}{r}} dr dT - dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Expanding,

$$ds^2 \approx -2 dr dT - dr^2 - r_S^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) + O(r - r_S)$$

Thus the metric is not diagonal near the Schwarschild radius; but it is perfectly smooth. For example, it remains invertible. At infinity, the metric reduces to Minkowski metric in spherical co-ordinates.

2.3. **There is still a singularity at $r = 0$.** This cannot be removed by any change of co-ordinates. For example, curvature scalars such as $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ diverge there. There is a point-like source located at that point.