Lecture 16

1. The Schwarzschild Solution

1.1. Now we can calculate the components of the Ricci tensor of a centrally symmetric metric. We don’t need the full Riemann tensor for now. Of more immediate interest to us are the components of the Ricci tensor. It is best to use the explicit formula

\[
R_{\mu\nu} = \frac{\partial \Gamma^\rho_{\mu\nu}}{\partial x^\rho} - \frac{\partial \Gamma^\rho_{\mu\nu}}{\partial x^\rho} + \Gamma^\rho_{\mu\nu} \Gamma^\sigma_{\rho\sigma} - \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\rho\nu}
\]

We leave it as an exercise to calculate these components. Setting them to zero will yield the vacuum Einstein equations.

1.1.1.

1.2. The centrally symmetric vacuum Einstein equations reduce to a system of ordinary differential equations.

\[
e^{-\lambda} \left[ \frac{\nu_r}{r} + \frac{1}{r^2} \right] - \frac{1}{r^2} = 0
\]

\[
e^{-\lambda} \left[ \frac{\lambda_r}{r} - \frac{1}{r^2} \right] + \frac{1}{r^2} = 0
\]

\[
\lambda_t = 0
\]

There is one more equation but it is not independent of these. So we see that \( \lambda \) is independent of time. This is the first half of Birkhoff’s theorem. But adding the other two equations give

\[
\lambda_r + \nu_r = 0
\]

so that

\[
\lambda + \nu = f(t)
\]

for some function of \( t \) alone. Now we pull out the fact that we still have a freedom to make the change of variables

\[
t \mapsto T(t)
\]

This will change the metric coefficients

\[
\nu \mapsto \nu + 2 \log \dot{T}, \quad \lambda \mapsto \lambda + 2 \log \dot{T}
\]

So by choosing \( T \) appropriately we can set \( f(t) \mapsto 0 \).
1.2.1. *All time dependence can be transformed away. This is Birkhoff’s theorem.*

Now we just have one independent variable that only depends on $r$. The equations is easily solved with the b.c. that $e^\lambda \to 1$ as $r \to \infty$.

$$(e^{-\lambda})_r + \frac{e^{-\lambda}}{r} = \frac{1}{r} \implies e^{-\lambda} = 1 + \frac{r_0}{r} = e^\nu$$

Here $r_0$ is a constant of integration with the dimensions of length, called the Schwarzschild radius.

1.2.2. *The constant can be fixed by comparison to Newton’s theory.* Recall that $g_{00} \approx 1 + 2\phi$ in the Newtonian approximation and that $\phi = -\frac{km}{r}$ where $k$ in Newton’s constant, where $m$ is the total mass of the centrally symmetric body. Comparing, we get

$$r_s = km$$

Putting in factors of $c$ explicitly (recall that $\frac{km}{r}$ has dimensions of the square of velocity)

$$r_s = \frac{km}{c^2}.$$ 

For the Sun, this is about 10 km. For the Earth, 5 cm.

1.3. *Thus we are led to the Schwarzschild solution to vacuum Einstein equations.*

$$ds^2 = \left(1 - \frac{r_s}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{r_s}{r}} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

Far from the center this is the metric of a star or mass $m$. As we get closer the departure from Newtonian gravity becomes larger. Soon we will solve the geodesic equation to get the corrections to Kepler’s laws.

1.4. *Something goes wrong as we approach the Schwarzschild radius.* The metric tensor is a diagonal matrix one of whose entries diverges as $r \to r_s$. For a long time this was a source of confusion. Upon careful examination we see that in fact the geometry itself is smooth; it is just that the co-ordinate system is breaking down at $r = r_s$. This is similar to the Euclidean metric of the plane in spherical polar co-ordinates at the origin:

$$ds^2 = dr^2 + r^2 d\theta^2$$

There is nothing special at $r = 0$, we see by transforming to Cartesian co-ordinates.

2. Painleve Co-ordinates

2.1. *We can make a change of co-ordinates that removes the divergent term at $r = r_s$.* We change just the time co-ordinate, a way that can depend on radius $t \mapsto f(T,r)$:

$$dt = f_TdT + f_r dr$$
\[ ds^2 = \left(1 - \frac{r_S}{r}\right) \left[ f_T dT + f_r dr \right]^2 - \frac{dr^2}{1 - \frac{r_S}{r}} - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \]

We can choose \( f \) such that the potential divergent term cancels:

\[ \left(1 - \frac{r_S}{r}\right) f_r^2 dr^2 - \frac{dr^2}{1 - \frac{r_S}{r}} = -dr^2 \]

Then we would have

\[ ds^2 = \left(1 - \frac{r_S}{r}\right) \left[ f_T^2 dT^2 + 2 f_r f_T drdT \right] - dr^2 - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \]

We can also choose \( f_T = 1 \) so that there are minimal changes to the other terms in the metric.

\[ ds^2 = \left(1 - \frac{r_S}{r}\right) \left[ dT^2 + 2 f_r dr \right] - dr^2 - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \]

\[ \left(1 - \frac{r_S}{r}\right) f_r^2 = -1 + \frac{1}{1 - \frac{r_S}{r}} \implies \left(1 - \frac{r_S}{r}\right) f_r = \pm \sqrt{\frac{r_S}{r}} \]

It turns out that the negative sign corresponds to the blackhole. We get

2.2. **The Schwarschild geometry in Painleve co-ordinate is smooth at** \( r = r_S \).

\[ ds^2 = \left(1 - \frac{r_S}{r}\right) dT^2 - 2\sqrt{\frac{r_S}{r}} drdT - dr^2 - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \]

Expanding,

\[ ds^2 \approx -2drdT - dr^2 - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) + O(r - r_S) \]

Thus the metric is not diagonal near the Schwarschild radius; but it is perfectly smooth. For example, it remains invertible. At infinity, the metric reduces to Minkowski metric in spherical co-ordinates.

2.3. **There is still a singularity at** \( r = 0 \). This cannot be removed by any change of co-ordinates. For example, curvature scalars such as \( R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \) diverge there. There is a point-like source located at that point.