GRAVITATION F10

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Lecture 2

1. Causality

1.1. **We will throughout use units in which the velocity of light is unity.** This means that length is measured in time units (e.g., light seconds).

1.2. **There are three kinds of vectors in Minkowski space: time-like, null or space-like.** Time-like vectors have $u \cdot u = u^T \eta u > 0$; null vectors have $u.u = 0$ and space-like vectors satisfy $u.u < 0$.

1.2.1. *By a Lorentz transformation any time-like vector can be brought to the form* $u = (u_0, 0, 0, 0)$. This is analogous to the way any vector in Euclidean space be made to point in the x^1-direction. Similarly any spacelikevector can be brought to the form $(0, u_1, 0, 0)$ and any null vector to the form $(a, \pm a, 0, 0)$.

1.3. **If we want to send a light signal from x to x', it is necessary that $x'_0 > x_0$ and $(x' - x) \cdot (x' - x) = 0$.** This means that the point x' lies to the future of x and the vector connecting them is null. Thus null vectors are also said to be light-like. These conditions are unchanged under proper Lorentz transformations.

1.3.1. *The sign of the zeroth component of a time-like or null vector cannot change under Lorentz transformations for which $\Lambda_{00} > 0$. But such transformations can change the sign of the zeroth component of a spacelike vector.* Thus, if two points in space-time that are separated by a space-like vector we cannot tell which is to the future. The only way to avoid contradictions (so that the future cannot affect the past) is to postulate that space-like separated points cannot comunicate with each other. This is equivalent to postulating that

1.4. **No particle or signal can move faster than light.** This is the principle of causality. A particle moving faster than light could be used to send a signal to a point in the past.

2. Relativistic Mechanics

2.1. **Energy and momentum transform together as a four-vector under Lorentz transformations.**

$$p = (E, cp_1, cp_2, cp_3).$$

2.2. **The relation between energy and momentum is.**

$$p \cdot p = m^2 c^4, \quad E > 0$$

Geometically, this one sheet of a hyperboloid in four-dimensional space, called the *mass shell*.

$$E = \sqrt{m^2 c^4 + c^2 \mathbf{p}^2}$$

In particular, even a particle at rest has energy

$$E = mc^2.$$

For velocities small compared to c,

$$E \approx mc^2 + \frac{\mathrm{p}^2}{2m}$$

The second term is the Newtonian formula for kinetic energy. Since the mass of particles usually do not change,in most situations we do can ignore the first term. But in nuclear reactions, this energy can be released with spectacular results.

2.3. **For massless particles the momentum vector is null.** The inner product of momentum with itself is zero

$$p \cdot p = 0.$$

Geometrically, the set of null momenta is a cone in four-dimensional space.

The relation of energy to momentum is

$$E = c|\mathbf{p}|$$

2.4. **Free particles move along straight lines in Minkowski space.** Massive particles move along time-like straight lines: the tangent vector has positive inner product with itself. Massless particles move along null lines.

2.5. **Conservation of energy-momentum places important restrictions on decays and scattering of elementary particles.**

2.6. **The path of a particle is given by a curve in space-time.** At any instant in time the particle has a position $\mathbf{x} \in R^3$. If we piece these positions together, we get a continuous curve in space-time. It is sometimes called its 'worldline' even thought in general it is not a line but can be curved. We don't have to think of the position as a function of time: it is often more convenient to think of both space and time as functions of some other variable τ. The derivative of these functions form a vector (the tangent vector to the curve). The principle of causality requires that the tangent vector to the worldline of a particle should be time-like or null, never space-like.

$$\frac{dx^\mu}{d\tau} \eta_{\mu\nu} \frac{dx^\nu}{d\tau} \geq 0$$

2.7. **A convenient parameter on a worldline is its arc-length. For time-like curves, it is the time measured by an observer attached to the line.**

$$s = \int_0^\tau \sqrt{\frac{dx^\mu}{d\tau} \eta_{\mu\nu} \frac{dx^\nu}{d\tau}} d\tau$$

2.8. **The tangent vector of a time-like curve parametrized by proper time is called four velocity. It is a vector of unit length.**

2.9. **The path of a free massive particle is the one that minimizes its arc-length or proper time.** Of course, this is a straighline. This variational principle will become important as a way to determine the path of particles in curved spacetimes.

2.10. **The energy-momentum of a particle is its mass times the four-velocity.**

$$p^\mu = m\frac{dx^\mu}{ds}.$$

The relation of energy to momentum follows from this:

$$p \cdot p = m^2.$$

Also, positivity of energy $p_0 > 0$ is the same as the condition that the curve be future pointed.

3. The Schrodinger equation has to be changed to take account of relativity

Recall that in quantum mechanics

$$\mathbf{p} = -i\hbar\frac{\partial}{\partial \mathbf{x}}, \quad E = i\hbar\frac{\partial}{\partial t}.$$

The relation

$$E = \frac{\mathbf{p}^2}{2m}$$

of non-relativistic mechanics gives the Schrodinger equation for a free particle:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial \mathbf{x}^2}.$$

For a relativistic particle instead

$$E^2 = c^2\mathbf{p}^2 + m^2c^4$$

leading to

3.1. **The Klein Gordon Equation.**

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial \mathbf{x}^2} - \left(\frac{mc}{\hbar}\right)^2\psi.$$

The quantity $\frac{\hbar}{mc}$ has the dimension of length; it is a fundamental property of a particle determined by its mass, called its Compton wavelength. It is called that because this combination first appeared in Compton's explanation of the scattering of gamma rays by elecrons. Now we know that this equation only describes spin zero particles. Dirac discovered the correct equation for spin one half particles like the electron.

3.2. **For a massless particle this becomes the wave equation.**

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial \mathbf{x}^2}$$