GRAVITATION F10

S. G. RAJEEV

Lecture 20

1. Gravitational Deflection of a Light Ray

1.1. **The path of a light ray is a null geodesic.** We can use the geodesic equation again, except that the parameter τ no longer has the meaning of arc length. In fact the Hamiltonian of the path vanishes

$$g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=0.$$

1.2. **For the Schwarschild metric, null geodesics also have conservation laws due to time translation and rotation invariance.** The orbit still lies in a plane. The plane can be chosen to be $\theta=\frac{\pi}{2}$. Thus we have for null geodesics

$$\frac{E^2}{1-u}-\frac{l^2u'^2}{1-u}-l^2u^2=0.$$

Or

$$l^2u'^2=E^2-l^2u^2+l^2u^3$$

We can also eliminate the constants by differentiating and writing this as

$$u''+u=\frac{3}{2}u^2$$

The nonlinear term is the GR correction. Treating it as small we get the solution, to first order

$$u\approx A\sin\phi+\frac{3}{4}A^2+\frac{A^2}{4}\cos 2\phi$$

where A is a constant of integration.

1.3. **Star light is deflected by the gravitational field of the Sun.** The closest approach D to the center occurs when u is a maximum $\frac{r_s}{D}$:

$$A\cos\phi + \frac{A^2}{4} 2\sin 2\phi = 0$$

A solution is $\phi = \frac{\pi}{2}$. That is

$$\frac{r_s}{D} = A + \frac{A^2}{2}$$

At infinite distance, $u = 0$ and the angle is given by

$$\sin\phi + \frac{3}{4}A + \frac{A}{4}\cos 2\phi = 0$$

Since A is small

$$\phi \approx -A \approx -\frac{r_s}{D}$$

to first order. Thus as the ray starts at infinity, passes near the center and emerges at infinite distance again, it is deflected through an angle

$$2\frac{r_s}{D}$$

If we are observing a ray of light from a star behind the Sun, the largest deflection will occur for rays that pass close to the Sun: D must be the radius of the Sun. Using $D = 7\times 10^5 km$ and $r_s = 3km$ gives a deflection of $\approx 10^{-5}\text{rad} \approx 2''$. This was big enough to observed in a famous expedition in 1919. Confirmation of it in later observations in 1922 and in modern times helped establish GR as a physical theory.

2. Gravitational Lenses

2.1. **If there is a heavy dark object near the line of sight to a star, there are two paths that light can take.** We follow the discussion in the expository article S. Refsdal Mon. Not. Roy. Soc. page 23(1964)

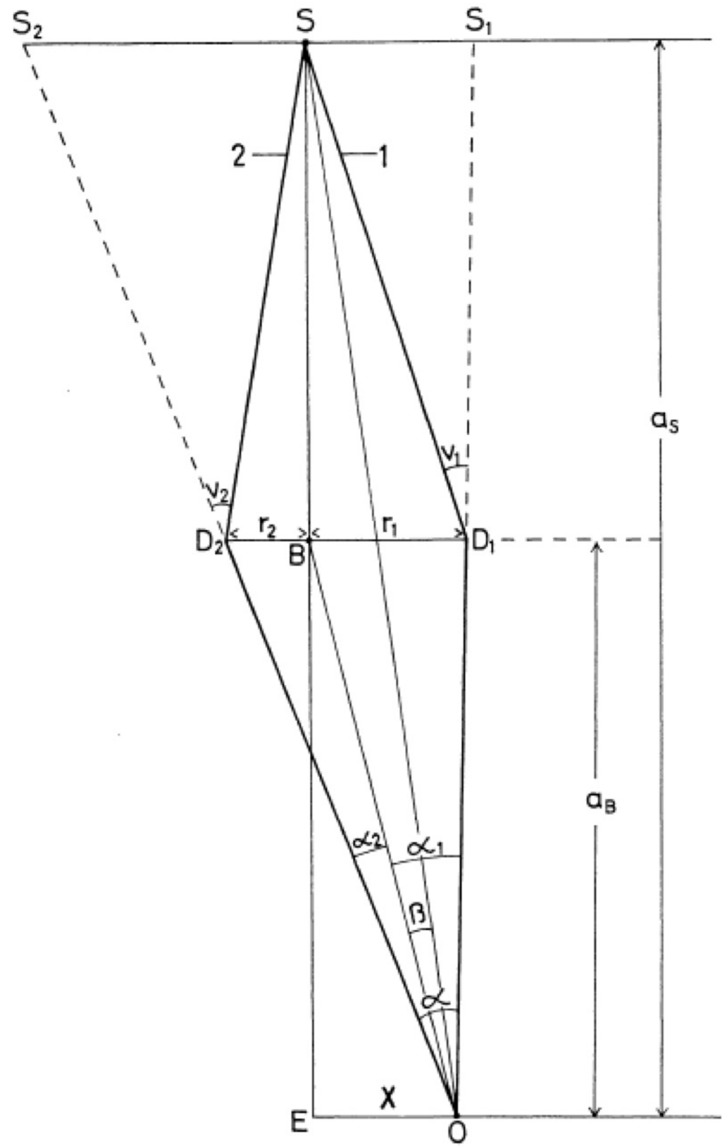

FIG. 1.—*The two light rays from S to O.*

Let the source be S, the object be at O. In between is a heavy dark object B . There is a plane containing SBO. There are two paths that light can take, on on each side of the line SO. Thus there will be two images S_1, S_2 formed: the observer at O will see two stars separated by an angle α as shown in the figure. Einstein himself noticed this in 1937 and calculated the intensities of the images. But he said that this is not likely to be of observational importance: if S and B are stars within our galaxy, the distances and masses are not big enough to produce interesting gravitational lenses. But at cosmological distances and galactic masses, the effect is spectacular.

2.1.1. *If B lies exactly on the line of sight, the image is a ring.* In this case there is circular symmetry around the line *SBO*. This is called the Einstein ring. The angular radius of the ring is given by the gravitational deflection angle.

2.1.2. *If B is close to, but not on, the line of sight there wll be multiple images.* We need some concepts from scattering theory in the geometrical optics approximation to develop a theory for them.

2.2. **Thinking of the angle of the outgoing light ray β as a function of the incoming angle α gives a transformation of the source sphere to the image sphere.** This point of view holds even without spherical symmetry of *B* nd even if the deflection is large.

2.3. **The intensity of the image of a point source is the inverse determinant of the Jacobian $\frac{1}{\det \frac{\partial\beta}{\partial\alpha}}$ of this transformation.** This follows from standard principles of geometrical optics.

2.4. **If this determinant vanishes the image will be bright.** The actual intensity will be finite because the source has finite angular size; also light has finite wavelength. The set of values of the source position α for which $\det \frac{\partial\beta}{\partial\alpha} = 0$ is a (possibly disconnected) curve on the sphere, called a *caustic*. If the source lies exactly on a caustic, there are an infinite number of images, on the *critical lines,* the values of β for which $\det \frac{\partial\beta}{\partial\alpha} = 0$ .If the source lies outside a caustic, there is only one image: there is no gravitational lensing. If we imagine moving the source position, each time it crosses a caustic, two new images are created. Thus there are always an odd number of images visible, including the original source.

2.5. **Because the different images correspond to different optical path lengths, there is a time delay in the signals.** The source intensity varies randomly with time. If we measure the apparent brightness of the images, they will match up to a shift in time. The difference in times of arrival can be related to the mass contained within the two optical paths.

2.6. **Gravitational lenses are an effective probe of the mass density of the universe.** By measuring the deflection angle and the time delay of quasars can be used to measure the mass density of the universe over cosmological distances. This adds to mounting evidence that

2.7. **Most of the mass in the universe is not due to stars or galaxies, but is some form of dark matter.**

3. Gravitational Red Shift

3.1. **The wavenumber and frequency of a photon transform together as a covariant vector.** This vector is null

$$g^{\mu\nu}k_\mu k_\nu = 0$$

$k_\mu = g_{\mu\nu}\frac{dx^\nu}{d\tau}$ is the covarant tangent vector to the null geodesic along which the light ray propagates.

3.1.1. *In a stationary space-time, k_0 is a constant along the light ray.* This follows from the variational principle for geodesics

$$S = \frac{1}{2}\int g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}d\tau$$

The geodesic equation has the form

$$\frac{d}{d\tau}\left[g_{\mu\nu}\frac{dx^\nu}{d\tau}\right] = \frac{1}{2}\partial_\mu g_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}$$

If $\partial_0 g_{\mu\nu} = 0$,

$$\frac{dk_0}{d\tau} = 0.$$

3.2. **The frequency of a photon of wave number k_μ is $\omega = u^\mu k_\mu$ where u^μ is the 4-velocity of the observer.** Recall that u^μ is the unit tangent vector to the worldline, which is always time-like. In flat space, a stationary observer has $u^\mu = (1,0,0,0)$and the frequency $\omega = k_0$, the time component. In general the frequency measured depends on the velocity of the observer (Doppler shift).

3.3. **The 4-velocity of a stationary observer in Schwarzschild space-time is $u^\mu = \left(\frac{1}{\sqrt{1-\frac{r_s}{r}}},0,0,0\right)$.** The Schwarzschild metric hasa prefered reference frame, in which the center is at rest. An observer who is at this rest in this frame has constant r,θ,ϕ. Hence $u^\mu = (u^0,0,0,0)$. In order to have unit length $g_{00}u^0u^0 = 1$. And of course, $g_{00} = 1-\frac{r_s}{r}$.

3.4. **A photon emitted by a stationary atom at r_1 will be seen to have a lower frequency when observed by a stationary observer at $r_2 > r_1$.** Recall that along a null geodesic, k_0 is constant. Thus

$$\omega_1 = \frac{k_0}{\sqrt{1-\frac{r_s}{r_1}}}, \quad \omega_2 = \frac{k_0}{\sqrt{1-\frac{r_s}{r_2}}}.$$

$$\omega_2 = \omega_1 \sqrt{\frac{1-\frac{r_s}{r_1}}{1-\frac{r_s}{r_2}}}$$

For distances large compared to $r_s = 2Gm$,

$$\omega_2 \approx \omega_1 \left(1 - \frac{Gm}{r_1} + \frac{Gm}{r_2}\right).$$

A phtoton emitted atom emitted at the surface of the Sun will have some natural frequency characteristic of that atom; if that photon is observed at the Earth its measured frequency will be lower by this factor compared to the same spectral line in a lab. (The Doppler shift due to the motion of the Earth should be taken into account, but that is easy.) Note that the redshift can be interpreted as the loss of "kinetic" energy by the photon as it climbs out of the graviiational field.

3.5. **This red shift was already measured in the 1920s, forming the last of the three classical tests of GR.** Soon after Einstein's GR Weyl proposed a beautiful unified theory of gravity and electromagnetism based on conformal ("gauge") symmetry. But it predicted the wrong value of the redshift; hence was ruled out as a physical theory. But this noble attempt by Weyl was the motivation for non-abelian gauge theories, which form the basis of particle physics today.

3.6. **The Mossbauer effect makes the measurement of gravitational red shift of a gamma ray emitted by an atomic nucleus in the Earth's gravitational field.** The effect is quite small, as the Schwarschild radius of the Earth is a few cm while the radius of the Earth is about $10^4 km$. Moreover, r_1 and r_2 can only be different by a few hundred meters. But the natural width of some nuclear gamma ray spectral lines are very small; so even a change of frequency of one in 10^{10} is measurable. The problem was that free nucleus will recoil as it emits the photon, and this causes an unpredictable shift in the frequency of the photon emitted. Mossbauer discovered that nucleii that are bound in a crystal do not suffer from this recoil broadening of spectral lines: the momentum is absorbed the whole crystal and not by a single nucleus. Hence it became possible to measure the gravitational red shift caused by Earth. (Pound and Rebka)