GRAVITATION F10

S. G. RAJEEV

Lecture 23

1. The De Sitter Metric

1.1. **The value of the cosmological constant is not determined by the principles of GR.** Einstein's equations with a cosmological constant is

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = k T_{\mu\nu}$$

Einstein himself wavered on whether Λ is zero or not; the original version of GR assumed that $\Lambda = 0$. But the principles on which GR is based (general covariance, equivalence) do not fix the value of Λ. There is no new symmetry in the theory if Λ were zero. In that sense assuming it is zero is not natural.

Indeed, it is the only new constant in GR, since k (built out of c and Newton's constant) is already known. Λ is a quantity with the dimensions of L^{-2}. The success of GR with $\Lambda = 0$ (e.g., perihelion test) shows that it must be small; more precisely, the length scale $\rho = |\Lambda|^{-\frac{1}{2}}$ must be larger than of the planetray orbits ($\sim 10^8$ km). The velocity distributions of stars in a galaxy do not agree with the predictions of GR and known (visible) sources of matter. But this discrepancy cannot be fit to a non-zero Λ; instead it appears to be due to a kind of matter that does not produce light ("dark matter"). Thus, for a long time it appeared that Λ was either zero or too small to be observed.

The cosmological constant goes by the more sensational name "dark energy" among science writers.

1.1.1. *Recent observations suggest that* $\Lambda > 0$. The situation has changed in the last decade with the observation of red shifts of distant galaxies and a measurement of distances to them. It appears now that at very large length scales GR can be correct only if Λ is small and positive: the distance ρ scale is several billion light years. Therefore for small distances of the size of the solar system , we can ignore Λ.

1.1.2. *The origin of the cosmological constant is a mystery.* We do not know why it is not zero; or why it is so small given that it is not zero. It is possible that the dark energy that supports de Sitter geometry has its origin in the vacuum energy of elementary particles. But all calculations give a value that is much bigger than the true value (by a fact of 10^{123}).

1.2. **The maximally symmetric solution of vacuum Einstein's equations with a positive cosmological constant is de Sitter space.** Thus even far away from all sources, space-time is not flat. Recall that the maximum number of independent Killing vectors is ten. Minkowski space is the maximally symmetric solution of Einstein's equation without a cosmolgical constant. When $\Lambda > 0$, the solution with the largest symmetry is de Sitter space. It is a space of constant curvature:

$$R_{\mu\nu\rho\sigma} = \lambda[g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}], \quad \lambda = -\frac{\Lambda}{3}$$

1.3. **For comparison, a sphere is also a space of maximal symmetry.** A sphere of radius ρ in n-dimensional Euclidean space is the solution to the equation

$$x_1^2 + \cdots x_n^2 = \rho^2$$

There are $\frac{n(n-1)}{2}$ independent rotations which are symmetries of this space.

It is a space of constant curvature, whose Riemann tensor is

$$R_{ijkl} = \rho[g_{ik}g_{jl} - g_{il}g_{jk}]$$

The metric of R^3 can be written as

$$ds^2 = dr^2 + r^2[d\theta_2^2 + \sin^2\theta_2 d\theta_1^2]$$

is spherical polar co-ordinates. Setting $r = 1$ we get the metric of the unit sphere S^2

$$ds_{S^2}^2 = d\theta_2^2 + \sin^2\theta_2 d\theta_1^2$$

More generally, the metric on S^m can be related to the metric on S^{m-2} by

$$ds_{S^m}^2 = d\theta_m^2 + \sin^2\theta_m ds_{S^{m-1}}^2$$

by recursively solving the constraint on the length of vectors.

1.4. **de Sitter space can be thought of as the pseudo-sphere.**

$$y_0^2 - y_1^2 - y_2^2 - y_3^2 - y_4^2 = -\rho^2$$

That is, it is the set of space-like vectors of length ρ in five-dimensional Minkowski space. The opposite sign leads to a different metric.

1.4.1. *Anti-deSitter space is the set of vectors of fixed length in $R^{2,3}$.*

$$y_0^2 + y_1^2 - y_2^2 - y_3^2 - y_4^2 = \rho^2$$

This space is of much interest to string theorists, but has limited use in cosmology.

1.4.2. *We can express the de Sitter metric in various co-ordinates.* A popular form is

$$ds^2 = dt^2 - e^{2\frac{t}{\rho}}(dx_1^2 + dx_2^2 + dx_3^2)$$

In these co-ordinates, the constant "time" slice is flat. This corresponds to solving the constraint above as follows:

$$y_0 = \rho\sinh\frac{t}{\rho} + \frac{r^2}{2\rho}e^{\frac{t}{\rho}}, \quad y_1 = \rho\cosh\frac{t}{\rho} - \frac{r^2}{2\rho}e^{\frac{t}{\rho}}, \quad y_{i+1} = e^{\frac{t}{\rho}}x_i$$

and $r^2 = \sum_i x_i^2$. This expression is particularly convenient as it allows us to think of of de Sitter space as a Lie group.

1.4.3. *The vectors* $e_0 = \frac{\partial}{\partial t}, \quad e_i = e^{\frac{t}{\rho}} \frac{\partial}{\partial x^i}$ *satisfy.*

$$[e_0, e_i] = \frac{1}{\rho} e_i, \quad [e_i, e_j] = 0$$

This is a solvable Lie algebra; the de Sitter space is its Lie group. The metric is induced by the quadratic form $e_0^2 - e_1^2 - e_2^2 - e_3^3$ through left-translations. In this point of view, energy and momentum do not commute unlike in Minkowski space. Instead, energy generates scale transformations in momentum. In the imit $\rho \to \infty$ we recover Minkowski space.

1.5. **The symmetry of de Sitter space is $SO(1,4)$.** This follows from the fact that the equation definiing it as a surface in five dimensional space is invariant under the five dimensional Lorentz group. The basis of infinitesimal de Sitter transformations can be thought of as 10 five by five matrices satisfying

$$[L_{ab}, L_{cd}] = \eta_{bc} L_{ad} - \eta_{ac} L_{bd} - \eta_{bd} L_{ad} + \eta_{ad} L_{bc}, \quad \eta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$