GRAVITATION F10

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Lecture 24

1. The Homogenous Isotropic Universe

1.1. **To a good approximation, the spatial metric of the universe can be taken to be homogenous and isotropic.** Homogenous means that every point is like any other: by an isometry we can map any point to any other point. Isotropy means that all the directions are the same. The subgroup of isometries that leave a point fixed will still rotate the tangent vectors at that point. Isotropy is the condition that tit leaves the length of vectors invariant. If we ignore "local" variations (average over a region that contains a few thousand galaxies) the universe looks the same at every point and in every direction.

1.2. **But the metric of space-time is not invariant under time translation; also under reflection in time.** It was a surprise in the early days of GR, when it predicted that the universe is expanding. Einstein even changed his thoery by adding a negative cosmological constant to get a static universe. But Hubble's discovery of red shift of galaxies showed that the original prediction was indeed correct.

1.3. **The spatial part of the metric can be a sphere, a hyperboloid or Euclidean.**

1.3.1. *If the spatial metric is a sphere, the total volume of space is finite: the universe is closed.Otherwise it is open.* These are the only (simply connected) manifolds of dimension three that are homogenous and isotropic.(If we only assume homogenity there are a few more.) It is useful to write the metric of these cases in a uniform manner so that we can study them together.

$$ds^2 = a^2 \left[\frac{dr^2}{1-\sigma r^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right], \quad \sigma = 1, -1, 0.$$

The parameter a sets the scale for the radius or equivalently, the curvature: it has the dimensions of length. $\sigma = 1$ is the sphere, $\sigma = -1$ is the hyperboloid and $\sigma = 0$ is just Euclidean space.

1.3.2. *If we use the radial distance as a co-ordinate the metric can be written in a more familiar form.* When $\sigma = 0$ we get the Euclidean metric in spherical polar co-ordinates.

For the sphere $\sigma = 1$, $d\chi = \frac{dr}{\sqrt{1-r^2}} \implies r = \sin\chi$

$$ds^2 = a^2 \left[d\chi^2 + \sin^2\chi \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right]$$

and for the hyperboloid $d\chi = \frac{dr}{\sqrt{1+r^2}} \implies r = \sinh\chi$

$$ds^2 = a^2 \left[d\chi^2 + \sinh^2\chi \left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

1.4. **When the spatial metric is homogenous and isotropic the space-time metric can be brought to the form.**

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-\sigma r^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

The radius can change with time. The co-ordinates in which the metric has this form are said to be "co-moving": the time axis through each point is a geodesic. At each point a stationary observer is freely falling .

1.5. **Observations suggest that $\sigma = 0$.** The spatial cross-section is flat. There is as yet no explanation of this fact.

1.5.1. *Inflation is the most natutal explanation for why the spatial metric is homogenous, isotropic and flat.* Inflation is the hypothesis that the universe expanded exponentially fast in the instants after the Big Bang, most likely due to a cosmological constant. A small region (the size of an elementary particle) of the original space expanded to form the whole of the universe we see today. Then, the initial inhomogenities and isotropies would not survive: all we see now are the values in a tiny region. There is as yet no solid evidence to support this idea, although it is theoretically attractive.

1.6. **The Ricci and Einstein tensors can now be calculated.** We get

$$G_0^0 = 3\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{\sigma}{a^2}\right], \quad G_j^i = \left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{\sigma}{a^2}\right]\delta_j^i, \quad G_i^0 = 0.$$

1.7. **The stress tensor of a homogenous and isotropic matter distribution has a simple form.**

$$T_\mu^\nu = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix}$$

Again ρ and p are constant in space but can depend on time. Conservation of energy-momentum gives

$$D^\mu T_\mu^\nu = 0$$

This implies

$$\frac{\partial}{\partial t}\left[a^3\rho\right] + 3pa^2\frac{da}{dt} = 0.$$

If we think $a^3 = V$ as an instantaneous measure of volume this is the first law of thermodynamics $dE + pdV = 0$. It is convenient to introduce the "Hubble constant" (constant in space but not in time)

$$H = \frac{\dot{a}}{a}$$

and write this as

$$\dot{\rho} = -3H(\rho + p)$$

1.7.1. *The equation of state of matter is a relation between density and pressure.* For example, in the case of massless particles (radiation, neutrinos etc.) the trace of the stress tensor is zero. (Recall that the stress tensor of Maxwell's theory is traceless.) Then

$$\rho = \frac{1}{3}p.$$

Galaxies or other weakly interacting non-relativistic matter can be thought of as the "dust particles" with $p = 0$. Even the cosmological constant (dark energy) can be thought of as a form of the stress tensor with $\rho = -p$. Thus radiation, dust and dark energy all have the equation of state

$$p = w\rho$$

with $w = \frac{1}{3}, 0, -1$ respectively. With this model

$$\dot{\rho} = -3(1 + w)\frac{\dot{a}}{a}\rho$$

so that

$$\rho = \rho_0 a^{-3(1+w)}.$$

In the special case of dust this is just conservation of mass. (There is no other form of energy in this case.)

1.8. **Einstein's equations become an ODE.** By looking at the combination $G_0^0 - G_i^i$ we get for example,

$$\frac{\ddot{a}}{a} = -\frac{k}{6}\left[\rho + 3p\right]$$

1.9. **We then get a simple equation for the rate of expansion.** For our simple model of matter above we can get, by simple calculus

$$-\frac{1}{H^2}\frac{dH}{dt} = \frac{3(1+w)}{2}, \quad H = \frac{\dot{a}}{a}.$$

Solving,

$$H = \frac{2}{3(1+w)t} \quad w \neq -1$$

and $H = \text{const}$ when $w = -1$.
leading to

$$a(t) = a_0 e^{Ht}, \quad w = -1$$

and

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}, \quad w \neq -1.$$

1.10. **When the universe is dominated by dark energy it grows exponentially fast.** This is believed to be the case in the very early universe: "inflation". The metric in this case is de Sitter.

1.11. **When it is radiation dominated the radius of the universe grows like the square root of time.** This is the case after the inflation is over, but before massive particles are created.

1.12. **When it is dominated by dust the radius grows as a power law.** This is the case for the current epoch.

1.13. **At very large distances it is found that the rate of expansion is exponential.** The true energy density in the current epoch is probably a sum of matter and dark energy. Even if the dark energy is small, at long distances the exponential dominates over the power law, so it will lead to an exponential expansion. At smaller distances, dark energy can be ignored and we get a power law expansion.