GRAVITATION F10

S. G. RAJEEV

Lecture 25

1. Riemann Normal Co-ordinates

1.1. **The geodesic emanating from a point provide a special co-ordinate system called the Riemann normal co-ordinates.** This system is closest to the Cartesian co-ordinate system in Euclidean space. Let M be a Riemannian manifold of dimension n , with metric tensor g and $p_0 \in M$ is a point on it. Given any unit vector v at p_0 we can solve the geodesic equation to find a curve passing through p_0 and v as the tangent there. Within some neighborhood $U \subset M$ of p_0 these geodesics will not cross: there will be a unique geodesic connecting p_0 to any $p \in U$. Let $s(p)$ be the arc-length of the geodesic from p_0 to p. We can then assign to this point the co-ordinate $s(p)v \in R^n$. This is the Riemann normal co-ordinate system.

1.2. **The metric tensor is the identity matrix and the Christoffel symbols vanish at the origin of a Riemann normal system.** Since the derivatives of g at a point are determined by Γ at that point, we get by the Taylor series expansion,

$$g_{\mu\nu}(x) = \delta_{\mu\nu} + \mathrm{O}(x^2)$$

The derivatives of Γ may not vanish even at the origin. In fact the Riemann tensor at the origin will be

$$R^{\mu}_{\nu\rho\sigma}(0) = \partial_\nu \Gamma^{\mu}_{\rho\sigma}(0) - \partial_\rho \Gamma^{\mu}_{\nu\sigma}(0)$$

We can express the second derivatives of the metric at a point in terms of the Riemann tensor at that point. By the Taylor series again,

$$g_{\mu\nu}(x) = \delta_{\mu\nu} - \frac{1}{3} R_{\mu\rho\nu\sigma}(0) x^\rho x^\sigma + \mathrm{O}(x^3)$$

These facts can be proven by expanding the metric in a Taylor series and also solving the geodesic equation as a power series and then matching coefficients. The radius of convergence of these series is the distance at which a pair of geodesics emanating from p_0 will collide.

1.3. **Thus up to a co-ordinate transformation, all Riemannian manifolds look like Euclidean space to second order in the distance.** This is the mathematical statement of the quivalence principle. Rather, it is because Riemannian geometry satisfies this property that it is the correct mathematical description of gravity.

1.4. **The Ricci tensor describes the rate of growth of the volume of a small sphere, compared to the volume of the same sphere in Euclidean geometry.** Recall that the colume density at a point is $\sqrt{\det g}$. (In Lorentzian signature we might out a negative sign under the root.) Also

$$\sqrt{\det\left[1+A\right]} = \exp\left[\frac{1}{2}\mathrm{tr}\log\left(1+X\right)\right]$$

$$= \exp\left[\frac{1}{2}\mathrm{tr}\left\{X - \frac{X^2}{2} + \frac{X^3}{3}\cdots\right\}\right]$$

$$= 1 + \frac{1}{2}\mathrm{tr}X + \frac{1}{4}\left[\left(\mathrm{tr}X\right)^2 - \mathrm{tr}X^2\right] + \cdots$$

$$\sqrt{\det g}(x) = 1 - \frac{1}{6}R_{\rho\sigma}x^\rho x^\sigma + \cdots$$

By comparison, we see that the volume of a sphere with smal radius is less than that of Euclidean space, when the Ricci tensor is positive (i.e., $R_{\mu\nu}u^\mu u^\nu > 0$ for all u) . Thus, for the sphere, the volume (actually area) of a disc of radius s is

$$2\pi\int_0^s \sin\theta d\theta \approx \pi\left[s^2 - \frac{s^4}{12} + \cdots\right]$$

Conversely, on spaces of negative Ricci tensor, the volume grows faster than in Euclidean space.

2. Myer's Theorem

2.1. **A space of positive Ricci tensor must have finite diameter.** This is a theorem of Myers. The diameter of a space is the largest distance between a pair of points in it. (A better name for it would have been diagonal, but everyone calls it diameter these days.) The proof uses the variational principle for geodesics. Myer's thoerem is a precursor to the singularity theorems of GR. These theorems of Penrose and Hawking say that a Lorentzian space-time whose Ricci tensor has positive time-like components cannot have arbirrary long time-like geodesics: a singularity because time does not last for ever. So it is worth learning Myer's theorem before learning this part of GR.

2.2. **Recall the action (also called energy by mathematicians) of a curve on a Riemannian manifold.**

$$S(\gamma) = \frac{1}{2}\int_0^l g(\dot\gamma, \dot\gamma)dt$$

We consider the space of curves starting at some point p and ending at some point q. By multiplying t by an appropriate constant, we can choose the length of the tangent $\dot\gamma$ at the starting point to be unity.

2.3. **Its first variation gives the geodesic equation.** Suppose v is some vector field defines in some neighborhood of γ. Then for small enough τ, we can define a new curve

$$\gamma_{\tau,v}(t) = \exp[\tau v]\gamma(t)$$

That is, go out a distance τ along the geodesic starting at $\gamma(t)$ and tangential to $v(\gamma(t))$ at that point. The first variation of S along v is

$$\left[\frac{dS(\gamma_{\tau,v})}{d\tau}\right]_{\tau=0} = \int_0^l g\left(D_{\dot\gamma}v, \dot\gamma\right) dt$$

If the first variation (keeping boundary values fixed $v(p) = v(q) = 0$) of this vanishes for all v, then γ is a geodesic. Then $\dot\gamma$ has constant length (unity) and

$$S = l$$

is just the arc-length. If the geodesic is also minimizing, then l is the distance between p and q.

2.4. **The second variation gives the Jacobi equation.** The second varation of the action functional is also of interest. A standard calculation (e.g., using Fermi co-ordinates) gives

$$\left[\frac{dS(\gamma_{\tau,v})}{d\tau}\right]_{\tau=0} = \int_0^l \left[g\left(D_{\dot\gamma}v, D_{\dot\gamma}v\right) - K(\dot\gamma\wedge v)\right] dt$$

where $K(u\wedge v)$ is the sectional curvature of the plane defined by u and v. If γ is a minimizing geodesic, this must be positive for all v :

$$\int_0^l \left[g\left(D_{\dot\gamma}v, D_{\dot\gamma}v\right) - K(\dot\gamma\wedge v)\right] dt \geq 0$$

2.4.1. *Now we make a variatonal ansatz.*

$$v(t) = V(t)\sin\frac{\pi t}{l}$$

where $V(t)$ is the parallel transport of some vector V at p. (That is, $D_{\dot\gamma}V = 0$). The factor of sin makes sure that v vanishes at the boundary. Then

$$\int_0^l \left\{|V|^2\left[\frac{\pi}{l}\right]^2\cos^2\frac{\pi t}{l} - K(\dot\gamma, V)\sin^2\frac{\pi t}{l}\right\} dt \geq 0$$

$$= \left\{\left[\frac{\pi}{l}\right]^2|V|^2 - K(\dot\gamma, V)\right\}\int_0^l \sin^2\frac{\pi t}{l}dt + |V|^2\left[\frac{\pi}{l}\right]^2\int_0^l\left\{\cos^2\frac{\pi t}{l} - \sin^2\frac{\pi t}{l}\right\} dt$$

The integral in the forst term is a positive quantity and the last term is zero. Hence for a minimizing geodesic,

$$\left[\frac{\pi}{l}\right]^2|V|^2 - K(\dot\gamma, V) \geq 0$$

This already gives Bonnet's theorem: if the sectional curvature is bounded $K(u, v) \geq k|u|^2|v|^2$ for some $k > 0$,

$$\left[\frac{\pi}{l}\right]^2 - k \geq 0$$

or

$$l \leq \frac{\pi}{\sqrt{k}}.$$

Recall that the Ricci tensor is the sum of sectional curvatures over an orthonormal frame

$$\mathrm{Ric}(u) = \sum_{i=1}^{n} K(u, e_i)$$

2.4.2. *We can do better by averaging the inequality over all V with a Gaussian measure $e^{-\frac{1}{\sigma^2}|V|^2}dV$ with some variance σ, subject to the constraint that $g(\dot{\gamma}, V) = 0$.* Since there are only $n-1$ independent such vectors, we will get

$$\left[\frac{\pi}{l}\right]^2 - \frac{1}{n-1}\mathrm{Ric}(\dot{\gamma}, \dot{\gamma}) \geq 0.$$

Recalling that $\dot{\gamma}$ is of unit length, we find that the length of a minimizing geodesic l must satisfy

$$\mathrm{Ric} \leq \left[\frac{\pi}{l}\right]^2 (n-1).$$

Thus the argument is essentially a use of the variational principle. More clever choice of variatonal ansatz gives improved versions of Myer's theorem.