GRAVITATION F10

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Lecture 27

1. The Cauchy Problem

1.1. **Given the initial value and time derivative of the field, there is a unqiue solution to the wave equation.**

1.1.1. *The solution can be expressed in terms of Green's functions.* We find first a solution to the wave equation with initial conditions

$$\Delta(x^0 = 0, \mathbf{x}) = \delta^3(\mathbf{x}), \qquad \frac{\partial \Delta}{\partial x^0}(x^0 = 0, \mathbf{x}) = 0$$

and another with initial conditions

$$\Delta_1(x^0 = 0, \mathbf{x}) = 0, \qquad \frac{\partial \Delta_1}{\partial x^0}(x^0 = 0, \mathbf{x}) = \delta(\mathbf{x}).$$

Then the solution to the wave equation with initial conditions

$$\phi(x^0 = 0, \mathbf{x}) = q(\mathbf{x}), \quad \dot{\phi}(x^0 = 0, \mathbf{x}) = p(\mathbf{x})$$

is given by

$$\phi(x) = \int \Delta(x^0, \mathbf{x} - \mathbf{y}) q(\mathbf{y}) d^3\mathbf{y} + \int \Delta_1(x^0, \mathbf{x} - \mathbf{y}) p(\mathbf{y}) d^3\mathbf{y}$$

1.2. **Using Fourier analysis we can get integral representations for these Green's functions.**

$$\Delta(x) = \int e^{i\mathbf{p}\cdot\mathbf{x}} \cos|\mathrm{p}| x^0 \frac{d^3\mathbf{p}}{(2\pi)^3}, \quad \Delta_1(x) = \int e^{i\mathbf{p}\cdot\mathbf{x}} \frac{\sin|\mathbf{p}|x^0}{|\mathbf{p}|} \frac{d^3\mathbf{p}}{(2\pi)^3}$$

It is also possible to evaluate these integrals and get certain explicit answers. The gist is that these functions are non-vanishing only along the light cone $x^2 = 0$. This is a form of Huyghen's principle: the solution to the wave equation in 3+1 dimensions only depnds on the data in the past light cone.

For more see e.g., Itzykson and Zuber Intro. Quantum Field Theory.

1.3. **The Cauchy problem is to determine the solution of a hyperbolic system of PDEs given initial conditions for the field and its time derivative.**

1.3.1. *The semi-linear equations are the simplest kind of non-linearity, as the non-lear part does nor depend on the derivative of the field.* As long as the nonlinearities are not too severe, the Cauchy problem has a unique regular solution. Although there may not be anymore an explicit solution, we can often construct the solution as a series, each term of which is a mutliple integral over space-time. An example are the semilinear equations

$$\Box\phi = \lambda\phi^{2p}, \quad p < 3$$

When the power is greater than 3 finite energy initial conditions can evolve into singularities in finite time. At the critical value $p = 3$ the solution is regular as long as a certain norm of the initial condition is not too large; if the norm is large there are singularities in the future. The behavior near the boundary between these two regines is especially interesting: you can get a kind of scale invariant behavior.

1.3.2. *Quasilinear equations are the next level of complexity; here the piece that involves the second order derivative is the same as for the wave equation.* An example is the wave map equation, also called the non-linear sigma model by physicists. This is the analogue of the wave equation, where the field takes values in a curved Riemannian manifold. More precisely, let $\phi : R^{1,3} \to N$ where N is a Riemannian manifold with metric h. The most interesting case, describing the π mesons of nuclear physics, is when $N = S^3$ with the standard metric. Then the wave map equation is the extremum of the action

$$S = \frac{1}{2}\int h_{ij}(\phi)\partial_\mu\phi^i\partial_\nu\phi^j\eta^{\mu\nu}\sqrt{-\det\eta}d^4x$$

Thus,

$$\Box\phi^i + \Gamma^i_{jk}\partial_\mu\phi^j\partial_\nu\phi^k\eta^{\mu\nu} = 0$$

where Γ^i_{jk} are the Christoffel symbols of h. As you can see, this reduces to the geodesic equation when we ignore space-dependence; and to the wave equation when when the target is flat. This equation has regular solutions in 1+1 dimensions; in 2+1 dimensions it is critical: solutions are regular or not depending on how big a certain norm of the solutions is. In 3+1 dimensions, it is “supecritical”: even finite energy solutions will led to a singularity in finite time. An explicit form for such a singular solution was found by Turok and Spergel in the case where the target is S^3, the case of interest in nuclear physics. In polar co-ordonates in space-time and on the target

$$ds^2_\eta = dt^2 - dr^2 - r^2\left[d\theta^2 + \sin^2\theta d\phi^2\right], \quad ds^2_h = du^2 + \sin^2 u\left[d\Theta^2 + \sin^2\Theta d\Phi^2\right]$$

the Turok-Spergel solution is

$$u = 2\arctan\left(\frac{r}{t - t_1}\right), \quad \Theta = \theta, \Phi = \phi$$

It is singular at $t = t_1$ and the initial condition at any previous time is regular.

1.4. **These examples show that a nonlinear system of PDEs may not always have regular solutions.**

1.5. **In the case of Einstein's equations, we know from the Penrose-Hawking theorems that initial conditions that contain trapped surfaces will lead to singularities.** Can initial conditions without singularities or trapped surfaces lead to first the creation of such surfaces, (i.e., blackholes) then a singularity? Can two blackholes collide and merge to form a bigger blackhole, while the singularities are hidden from view of external observers? How much gravitational radiation is emitted when blackholes collide? All these questions have answers hidden inside Einstein's equations. Unlocking these answers requires either numerical solutions or a powerful mix of analysis and geometry.

2. Geometric Analysis of Einstein's Equations

See P. T. Chruściel, G. J. Galloway and D. Pollack Bull. Amer. Math. Soc. 47 (2010), 567-638 for a recent review.

2.1. **Choquet-Bruhat pioneered the mathematial analysis of vacuum Einstein's equations.** The essential problem with Einstein's equations is not only that they are non-linear, but that they are underdetermined: the equations are unchanged under arbitrary co-ordinate transformations. So although there are 10 components to $g_{\mu\nu}$ there are only six independent equations. The remaining four degrees of freedom describe only the choice of co-ordinate system. Choquet-Bruhat advocated a particular co-ordinate system ('harmonic' co-ordinates) in which you could at least write Einstein's equations as a system of hyperbolic PDEs. It makes sense then to ask if initial conditions sufficiently close to Minkowski space (i.e., at a finite distance away from the trivial solution in some norm in the space of initial data) evolve regularly: physically you would expect gravitational waves to be emitted and after a long enough time, the metric tends to the Minkowski metric. This would be the statement that Minkowski space is globally stable even including the nonlinearities.

2.2. **Christodoulu and Kleinermann established global nonlinear stability of Minkowski space in GR without sources.** A tricky part is to discover the correct notion of a norm in the space of initial data which is preserved by time evolution. They use a kind of analogue of the stress tensor for gravity, a fourth rank tensor called the Bel-Robinson tensor to measure the departure from flat space; the norm is a weighted integral of this tensor. When this norm is small, the space-time has approximate Killing vectors leading to approximate conservation laws which, through an enormous bootstrap argument is used to establish that the solution is regular for ever.

2.2.1. *Later work of Rodniansky has simplified many of the proofs and strengthened these results.*

2.3. **Christodoulu has also shown that a spherically symmetric self-gravitating scalar field will collapse and form a black hole without forming naked singularities.** A massless scalar field is a good enough model for matter at very high densities. Using spherical symmetry we can eliminate the gravitational degrees of freedom and write the equations as a (very non-linear) system for the scalar field alone. Christodoulu combines insights from geometry, physics and analysis to show that although singularities can occur, they will be invisible from afar.

2.4. **It remains open whether cosmic censorship holds in general.** It is believed that under stable regular initial conditions, the singularities of GR are invisible to an observer at infinity.

2.5. **It is believed that the stationary solutions of vacuum Einstein's equations are fully determined by mass and angular momentum.** Higher moments are believed to not matter, a total contrast from Maxwell's theory. This "no hair" theorem has been proved under the assumption that the space-time is analytic: but this is too strong.

2.6. **It is believed that the area of event horizons will always grow with time.** This is the second law of blackhole dynamics, analogous to thermodynamics. The area of a blackhole is proportional to its entropy, an argument of Hawking and Beckenstein suggests. To establish this irreversibility without assuming a static background would be the analogue of non-equilibrium thermodynamics in GR. This is still not possible.

3. Numerical Solution of Einstein's Equations

3.1. **Pretorius and others have made enormous progress towards numerical solution of Einstein's equations.** We understand complicated hydrodynaic phenomena numerically. This powerful tool was missing in GR until recently. All attempts to solve interesting problems (e.g., collision of blackholes) ended in a singularity. But this was nota true singularity of the geometry, but of the co-ordinate system. For example, the harmonic co-ordinates of Choquet-Bruhat break down. Ingenious ideas were invented to circumvent this. Pretorius found a modification of the harmonic co-ordinates that would evolve the equations in a numerically stable way. Campanelli et. al. found another method as well. So now it is possible to solve Einstein's equations in interesting highly nonlinear situations. An exciting future for an ancient subject.