GRAVITATION F10

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Lecture 4

1. Variational Principles

1.1. **The fundamental laws of classical physics are differential equations which arise from variational principles.**

1.1.1. *Not all differential equations follow from variational principles. It is a significant fact that those that arise in fundamental physics do.* The equation of a damped harmonic oscillator does not follow from a variational principle.

1.1.2. *A variational principle is the condition that some function be unchanged under small variations in its arguments.* Thus, the function is a minimum or maximum or more commonly, it is just an extremum. Could be a saddle or inflection point. Variational principles used to be called "principles of least action" but that terminology is misleading: in most cases of physical interest, we do not get a minimum:the first order variation vanishes but the second variation is not usually positive.

1.2. **The condition that a function of n variables be an extremum is that the derivatives vanish.** We will only consider smooth functions.

$$\delta f = \frac{\partial f}{\partial x^i}\delta x^i = 0, \forall \delta x^i \implies \frac{\partial f}{\partial x^i} = 0.$$

Suppose we have n equations on the variables x^i

$$E_i(x) = 0.$$

Under what conditions do these follow from the condition that some function be an extremum? For $E_i = \partial_i f$, it is necessary that

$$\frac{\partial E_i}{\partial x^j} - \frac{\partial E_j}{\partial x^i} = 0.$$

Thus not all systems of equations follow from such a condition.

Remark 1. If we regard $E_i dx^i = E$ as a 1-form, the condition that the set of equations $E = 0$ extremizes a function is $E \wedge dE = 0$. I will put technical comments in remarks likes this; you don't need to understand them to follow the rest of the lecture.

1.3. **The action of a classical trajectory is the integral of the kinetic energy minus the potential energy.** The action is a function of an infinite number of variables: the value of the function at any point is an independent variable.

$$S = \int L dt, \quad L = \frac{1}{2} m\dot{x}^2 - V(x)$$

1.4. **The condition that the action be an extremum (under variations that preserve the endpoints) gives the Newtonian equations of motion.** Read Landau and Lifshitz *Mechanics* for more complete discussion.

$$\delta S = \int_{t_1}^{t_2} \left[m\dot{x}\delta\frac{dx}{dt} - \frac{\partial V}{\partial x}\delta x \right] dt$$

But, the variation of the derivative is the derivative of the variation:

$$\delta\frac{dx}{dt} = \frac{d}{dt}\delta x$$

(Prove this using Fourier analysis).

$$\delta S = \int_{t_1}^{t_2} \left[m\dot{x}\frac{d\delta x}{dt} - \frac{\partial V}{\partial x}\delta x \right] dt$$

We can do an integration by parts

$$\delta S = \int_{t_1}^{t_2} \left[-m\ddot{x}\delta x - \frac{\partial V}{\partial x}\delta x \right] dt + \left[m x\dot{\delta x} \right]_{t_2}^{t_2}$$

Since the varation vanishes at the endpoints

$$\delta x(t_1) = \delta x(t_2) = 0$$

we get

$$\delta S = \int_{t_1}^{t_2} \left[-m\ddot{x} - \frac{\partial V}{\partial x} \right] \delta x \; dt = 0$$

But this is true for every variation. So it must be that

$$m\ddot{x} = -\frac{\partial V}{\partial x}$$

These are Newton's equations.

1.5. **A free particle moves along straight lines in Euclidean space.**

$$\delta \int \dot{x}_i \dot{x}_i dt = 0 \iff \ddot{x}_i = 0.$$

For given beginning and end points a straightline connecting them is the path of least action. Although the action is not the same as the length for all paths, at the minimum the are the same. Prove these statements.

1.6. **A relativistic mechanics the action of a free particle is.**

$$S = m\int \eta^{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}ds$$

For physical reasons we require that the tangent vector to the path (the four-velocity) to be not space-like.

$$\eta^{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} \geq 0.$$

The extremum occurs when

$$\frac{d^2x^\mu}{ds^2} = 0.$$

i.e., a straightline.

1.6.1. *This is only an extremum. A path that connects two points can have smaller action (and arc-length) if it has a segment that has very small* $\eta^{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}$*: where the particle is moving at a speed close to light.*

1.7. **The Lorentz force equation follows from a variational principle as well.**

$$S = m\int \eta_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}ds - \int qA_\mu(x)\frac{dx^\mu}{ds}ds$$

This leads to

$$m\eta_{\mu\nu}\frac{d^2x^\nu}{ds^2} = qF_{\mu\nu}\frac{dx^\nu}{ds}$$

Prove this, using

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

1.8. **Note that the action is unchanged under gauge transformations, except for boundary terms .** As $A_\mu \to A_\mu + \partial_\mu\Lambda$, using the chain rule of differentiation

$$\frac{d\Lambda}{ds} = \frac{\partial\Lambda}{\partial x^\mu}\frac{dx^\mu}{ds}$$

we get

$$S \to S - \int q\frac{d\Lambda}{ds}ds$$

the change is a total derivative: by integration by parts it depends only the endpoints. But endpoints are held fixed so this change does not matter in the variational principle.