GRAVITATION F10

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Lecture 5

1. The Principle of Equivalence

1.1. **All particles have the same acceleration in a gravitational field.** By a particle we mean here a body whose mass and size are small. Thus, the Earth is a particle compared to the Sun: we are ignoring tidal forces that depend on the finite size of the Earth. At first you might think that a feather and a baseball fall at different rates in a gravitational field. But if we were to eliminate friction and buoyancy forces, they will drop at the same rate. The fact that all particles react in the same way to the gravitational field make it very different from all other forces of nature; e.g., electromagnetic acceleration depends on the ratio of electric charge to mass.

1.1.1. *This is called the (weak) principle of equivalence.*

1.1.2. *Thus two observers at infinitesimally close points will move at constant velocity relative to each other if subjected only to gravity.*

1.2. **It is not possible to distinguish the effect of gravity from that of an accelerated reference frame, using local observations.**

1.2.1. *This is called the strong principle of equivalence.* Imagine that you are in a spaceship with no windows. From the fact that you feel no weight you cannot conclude that you are not in a gravitational field. If you turn on the rocket engine you will feel that you are pushed back in your seat: that does not mean you turned on a gravitational field either. If you open the windows, and made measurements of positions of distant stars for a long enough time, you can tell if you are orbiting a planet or star. Or , you might have another spaceship not too far away and you will see that over time your relative velocity has changed.

1.3. **The equivalence principle has been tested with increasing precision over the years.** According to legend, it was Galieleo that first demostrated this property of gravity by dropping an iron ball and a brass ball from the leaning tower of Pisa. Very precise tests were performed in the nineteenth century by Eotvos. So tests of the equivalence principle are still called Eotvos experiments. People have been looking in vain for small discrepancies for more than a century. Every time a deviation is discovered, it has disappered on closer inspection.

1.4. **A theory of gravity must be invariant under changes of reference frames that are accelerated relative to each other.**

1.4.1. *This is the principle of general relativity.* The theory of relativity originally only allowed inertial observers: those moving at constant velocity relative to distant stars. We say that the laws of electromagnetism and mechanics are covariant (both sides of the equation change the same way) under Lorentz transformations, which change from one inertial observer to another. We now postulate covariance under arbitrary changes of reference frames.

1.5. **A reference frame is a co-ordinate system in space-time.** Each observer assigns a value of time and position for a point in space-time: mathematically, this is a co-ordinate system in space-time. Inertial observers are related by linear changes of co-ordinates.

1.5.1. *Lorentz transformations are linear transformations among co-ordinates in space-time.* They can be represented by matrices

$$y = \Lambda x, \quad y^\mu = \Lambda^\mu_\nu x^\nu$$

In addition to being linear they must satisfy

$$\Lambda^T \eta \Lambda = \eta.$$

2. Constant Acceleration

2.1. **Accelerated reference frames are related to each other by non-linear co-ordinate transformations.** We will now look at the simplest case of an observer moving at constant acceleration.

2.2. **The trajectory of a particle moving with constant acceleration is a hyperbola.** Consider just one direction of space and time: only two dimensions of space-time for simplicity. Think of the velocity as a function of proper time of an electric charge moving in a constant electric field. It satisfies the differential equation

$$\frac{du^0}{d\tau} = au^1, \quad \frac{du^1}{ds} = au^0$$

where a is the constant acceleration. The solution is

$$u^0 = \cosh a\tau, \quad u^1 = \sinh a\tau$$

We choose the origin of τ so that the particle is at rest when $\tau = 0$. Also, since τ is the time measured by the observer following this trajectory, the velocity is the same as the unit tangent vector:

$$u^\mu = \frac{dx^\mu}{d\tau}$$

Solving again

$$x^0 = X^0 + \frac{1}{a}\sinh a\tau, \quad x^1 = X^1 + \frac{1}{a}\cosh a\tau$$

This is the equation for a hyperbola:

$$(x^1 - X^1)^2 - (x^0 - X^0)^2 = a^{-2}.$$

Analogous to the equation for a circle, but with a sign flipped: the acceleration is like the inverse of the radius.

2.3. **As $\tau \to \infty$ the trajectory is asymptotic to a null line.** The velocity tends to the velocity of light

$$\frac{dx^1}{dx^0} = \tanh a\tau \to 1$$

2.4. **The observer moving at constant acceleration cannot recieve signals from the points to the future of this asymptote.** A signal recieved by the observer at some instant will come from within the past light cone at that point on its trajectory. These light cones tilt as the velocity increases, so that they never includes points on the other side of the asymptote.

2.4.1. *This is an example of a horizon. Unruh showed that in the quantum theory, the accelerated observer will see thermal radiation coming from the horizon. Since he cannot make observations on the other side, he will observe that space-time has entropy.* Sadly, these deep ideas are outside the scope of this course.

Problem 1. Are there points in space-time to which the observer at constant acceleration cannot send signals?

3. Polar Co-ordinates

3.1. **Polar co-ordinates on the Euclidean plane are given by.**

$$x^1 = r\cos\phi, \quad x^2 = r\sin\phi$$

3.1.1. *The co-ordinate axes are circles (r = constant) and half-lines (ϕ =constant).*

3.1.2. *They break down along the half-line $x^2 = 0, x^1 > 0$: it is where ϕ jumps from 0 to 2π.*

3.1.3. *The distance between two neighboring points is.*

$$ds^2 = (dx^1)^2 + (dx^2)^2 = dr^2 + r^2 d\theta^2.$$

3.2. **The analogue of polar co-ordinates in Minkowski space is.**

$$x^0 = \rho\sinh\theta, \quad x^1 = \rho\cosh\theta$$

The distance between infinitesially close points in space-time is

$$ds^2 = \left(dx^0\right)^2 - \left(dx^1\right)^2 = \rho^2 d\theta^2 - d\rho^2$$

The co-ordinate axis with ρ =constant is a hyperbola: the trajectory of an observer with acceleration ρ^{-1}. The case $\rho = 0$ gives the light cones. This co-ordinate system only covers the region of Minkowski space with

$$\left(x^1\right)^2 \geq \left(x^0\right)^2.$$

Problem 2. Why?

This family of observers with constant accelerations cannot see outside of this region bounded by a light cone. To cover all of space-time we will need several such co-ordinate systems covering overlapping regions.