Lecture 6

1. General Co-ordinates

1.1. The choice of co-ordinate system should be adapted to the system being studied. For example, curvilinear co-ordinates are useful in solving the Laplace equation in various geometries.

1.2. The transformation between co-ordinates must be smooth functions. Within a region where both co-ordinate systems are valid, the transformation between them must be differentiable and invertible. A simple example is the transformation between cartesian and polar co-ordinate systems. More generally, the new co-ordinate system $x'^\mu$ is specified as a set of functions of the old co-ordinates $x^\mu$.

1.3. The gradient of a scalar field transforms as.

$$\frac{\partial f}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial f}{\partial x^\nu}$$

This is the chain rule of differentiation. Notice that the index $\nu$ is summed over. Another way to understand this transformation law is that the infinitesimal change in the scalar field is the same in both co-ordinate systems:

$$df = dx^\mu \frac{\partial f}{\partial x^\mu} = dx'^\mu \frac{\partial f}{\partial x'^\mu}$$

1.4. If the derivatives of a function vanish at a point in one co-ordinate system, they vanish in any co-ordinate system. The derivatives along the different co-ordinate axes of a function can be thought of as the components of a vector field. More generally,

1.5. The components of a vector field change under co-ordinate transformations in a similar way:

$$\omega'_\mu = \frac{\partial x'^\nu}{\partial x^\mu} \omega_\nu.$$
1.5.1. More precisely, fields that transform this way are called covariant vector fields. We will soon see another kind of vector called a contravariant (contra-gradient) vector that transforms oppositely.

1.5.2. Not every covariant vector field is the derivative of a function: the integrability condition is.

\[ \partial_\mu \omega_\nu - \partial_\nu \omega_\mu = 0 \]

**Problem 1.** Show that the above integrability condition is independent of changes of co-ordinates.

1.6. **The second derivatives of a function do not transform as a tensor.** More precisely, the second derivatives might vanish in at some point in one system but not in another.

By repeated use of the chain rule of differentiation,

\[ \frac{\partial^2 f}{\partial x^\rho \partial x^\mu} = \frac{\partial x^\sigma}{\partial x^\rho} \frac{\partial x^\nu}{\partial x^\mu} \frac{\partial^2 f}{\partial x^\sigma \partial x^\nu} + \frac{\partial x^\sigma}{\partial x^\rho} \frac{\partial x^\mu}{\partial x^\sigma} \frac{\partial x^\sigma}{\partial x^\nu} \frac{\partial^2 f}{\partial x^\nu}. \]

If the change of co-ordinates is linear, the last term vanishes; in general it won’t be zero. Thus we will need some new notion of derivative to go beyond the first derivative of a function.

1.7. **The tangent vector to a curve transforms as a contravariant vector: opposite to the gradient of a function.** A curve is given by specifying the co-ordinates as a function of some parameter \( x^\mu (\tau) \). The components of the tangent vector are \( \frac{dx^\mu}{d\tau} \). If we transform to some new co-ordinates

\[ \frac{dx'^\mu}{d\tau} = \frac{\partial x'^\mu}{\partial x^\nu} \frac{dx^\nu}{d\tau}. \]

More generally, a vector whose components that transform this way is called a contravariant vector:

\[ v'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} v_\nu \]

Remember that the derivatives of \( x \) w.r.t. \( x' \) form the inverse matrix to the derivative of \( x' \) w.r.t. \( x \).

\[ \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x^\nu}{\partial x'^\mu} = \delta_\rho^\nu. \]

Thus contravariant and covariant vectors transform opposite to each other.

1.7.1. **The Kronecker delta are components of the identity matrix.**

\[ \delta^\mu_\nu = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases} \]
1.8. The sum of the products of corresponding components of a covariant vector and a contravariant vector is a scalar: unchanged under co-ordinate transformations.

\[ \omega'_{\mu} = \frac{\partial x'_{\nu}}{\partial x_{\mu}} \omega_{\nu}, \quad v'^{\mu} = \frac{\partial x'_{\mu}}{\partial x_{\rho}} v^{\rho} \]

\[ \implies \omega'_{\mu} v'^{\mu} = \omega_{\nu} \frac{\partial x'_{\nu}}{\partial x_{\rho}} \frac{\partial x'_{\mu}}{\partial x_{\rho}} v^{\rho} = \omega_{\nu} \delta_{\nu}^{\rho} v^{\rho} = \omega_{\nu} v^{\nu}. \]

We took care that an index appears at most twice in a factor. Also, a pair of repeated indices can be replaced by another without changing the value:

\[ \omega_{\nu} v^{\nu} = \omega_{\mu} v^{\mu}. \]

2. The Metric in Curvilinear Co-ordinates

2.1. The infinitesimal distance \( ds \) between two neighboring points in Euclidean space in Cartesian co-ordinates is given by.

\[ ds^2 = \delta_{\mu\nu} dx^{\mu} dx^{\nu} \]

2.2. In a general co-ordinate system.

\[ ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} \]

where \( g_{\mu\nu} \) can depend on \( x^{\mu} \).

2.3. The components transform under changes of co-ordinates as.

\[ g'_{\rho\sigma} = \frac{\partial x'_{\mu}}{\partial x^{\rho}} \frac{\partial x'_{\nu}}{\partial x^{\sigma}} g_{\mu\nu} \]

To see this we have to remember that \( ds \) itself is independent of the co-ordinate system; and use the rule for each factor \( dx'_{\mu} = \frac{\partial x'_{\mu}}{\partial x^{\rho}} dx^{\rho} \).

2.3.1. We say that \( g_{\mu\nu} \) are the components of the metric tensor. Metric refers here to a measure of distance.

2.4. We can calculate \( g_{\mu\nu} \) by transforming from the Cartesian co-ordinate system or by some more direct geometrical argument.

- In the polar co-ordinate system of the plane \( ds^2 = dr^2 + r^2 d\phi^2 \)
- If we define \( x^\pm = \frac{x^0 \pm x^1}{\sqrt{2}} \), in Minkowski space \( ds^2 = 2 dx^+ dx^- \). In this case the metric tensor is not diagonal.
- The metric of \( R^3 \) in spherical polar co-ordinates is

\[ ds^2 = dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]
• For a more perverse example we note the prolate spheroidal co-
ordinates in Euclidean space $R^3$ which is useful in solving some
differential equations:

$$x^1 = \sinh r \sin \theta \cos \phi, \quad x^2 = \sinh r \sin \theta \sin \phi, \quad x^3 = \cosh r \cos \theta$$

$$ds^2 = \left[ \sinh^2 r + \sin^2 \theta \right] [dr^2 + d\theta^2] + \sinh^2 r \sin^2 \theta d\phi^2$$

• More examples can be found in the monograph of Morse and Fesh-
bach [1]. Or more conveniently on wikipedia these days.

REFERENCES