GRAVITATION F10

S. G. RAJEEV

Lecture 8

1. The Geodesic Equation

1.1. **Riemann discovered the essential features of metric geometry in arbitrary dimensions.** The key idea is that the distance between nearby points

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

contains all the essential information. In particular it determines the geodesics. The rate of divergence of nearby geodesics determines the curvature: how much the space differs from Euclidean space locally.

1.2. **A geodesic is an extremum of the action on the set of curves.**

$$S = \frac{1}{2} \int g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} d\tau$$

Again we stress that a geodesic is not always a minimum; also S is a convenient proxy for the more intuitive concept of distance. It can be shown that the extrema of and of the length are the same.

1.3. **This leads to a differential equation.**

$$\frac{d}{d\tau} \left[g_{\mu\nu} \frac{dx^\nu}{d\tau} \right] - \frac{1}{2} \partial_\mu g_{\rho\nu} \frac{dx^\rho}{d\tau} \frac{dx^\nu}{d\tau} = 0.$$

Straightforward application of the Euler-Lagrange equation

$$\frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu} = 0$$

with Lagrangian

$$L = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$$

$$\frac{\partial L}{\partial \dot{x}^\mu} = g_{\mu\nu} \dot{x}^\nu$$

1.4. **An equivalent formulation is.**

$$\frac{d^2x^\sigma}{d\tau^2}+\Gamma^\sigma_{\rho\nu}\frac{dx^\rho}{d\tau}\frac{dx^\nu}{d\tau}=0,\quad \Gamma^\sigma_{\rho\nu}=\frac{1}{2}g^{\sigma\mu}\left[\partial_\rho g_{\mu\nu}+\partial_\nu g_{\mu\rho}-\partial_\mu g_{\rho\nu}\right]$$

The $\Gamma^\mu_{\rho\nu}$ are called Christoffel symbols. Calculating them for some given metric is one of the joys of Riemannian geometry; an even greater joy is to get someone else to do the calculation.

Proof. Rewrite the geodesic equation as

$$g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2}+\partial_\rho g_{\mu\nu}\frac{dx^\rho}{d\tau}\frac{dx^\nu}{d\tau}-\frac{1}{2}\partial_\mu g_{\rho\nu}\frac{dx^\rho}{d\tau}\frac{dx^\nu}{d\tau}=0$$

where we use the chain rule to calculate $\frac{dg_{\mu\nu}}{d\tau}=\partial_\rho g_{\mu\nu}\frac{dx^\rho}{d\tau}$. Now, by averaging over the interchange of ρ,ν we get

$$\partial_\rho g_{\mu\nu}\frac{dx^\rho}{d\tau}\frac{dx^\nu}{d\tau}=\frac{1}{2}\left[\partial_\rho g_{\mu\nu}+\partial_\nu g_{\mu\rho}\right]\frac{dx^\rho}{d\tau}\frac{dx^\nu}{d\tau}$$

and

$$g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2}+\frac{1}{2}\left[\partial_\rho g_{\mu\nu}+\partial_\nu g_{\mu\rho}-\partial_\mu g_{\rho\nu}\right]\frac{dx^\rho}{d\tau}\frac{dx^\nu}{d\tau}=0$$

Now we get rid of the $g_{\mu\nu}$ by multiplying throughout by its inverse, using $g^{\sigma\mu}g_{\mu\nu}=\delta^\sigma_\nu$:

$$\frac{d^2x^\sigma}{d\tau^2}+\frac{1}{2}g^{\sigma\mu}\left[\partial_\rho g_{\mu\nu}+\partial_\nu g_{\mu\rho}-\partial_\mu g_{\rho\nu}\right]\frac{dx^\rho}{d\tau}\frac{dx^\nu}{d\tau}=0$$

□

1.5. **Given an initial point P and a vector V at that point, there is a geodesic that starts at P with Vas tangent.** This just follows from standard theorems about the local existence of solutions of ODEs. The behavior for large τ can be complicated: geodesics are chaotic except for metrics with a high degree of symmetry.

Remark 1. The following are more advanced points that you will understand only during a second reading , after you have already learned some Riemannian geometry.

1.6. **On a connected manifold, any pair of points are connected by at least one geodesic.** Connected means that there is a continuous curve connecting any pair of points.(To define these ideas precisely we wil need first a definition of a manifold, which we will postpone for the moment.) Typically there are several geodesics connecting a pair of points. For example of the sphere, there are at least two for every (unequal) pair of points: one direct route and that goes around the world.

1.7. **Curves minimizing the action and the length are the same.** This can be proved using a trick using Lagrange multipliers. First of all, we note that the length can be thought of the minimum of

$$S_1=\frac{1}{2}\left[\int g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\lambda^{-1}d\tau+\int\lambda d\tau\right]$$

over all non-zero functions λ. Minimizing gives $\lambda^{-2}\dot{x}|^2=1\implies\lambda=|\dot{x}|$. At this minimum $S_1[x]=l[x]$. Now S_1 is invariant under changes of parameters

$$\tau \to \tau'(\tau), \lambda' = \lambda \frac{d\tau}{d\tau'}$$

Choosing this parameter to be the arc length, S_1 reduces to the action. Thus they describe equivalent variational problems. Moreover, at the minimum S, S_1, l all agree.

1.8. **The shortest length of all the geodesics connecting a pair of points is the distance between them.** It is a deep result that such a minimizing geodesic exists. Most geodesics are extrema.

1.9. **Gauss and Riemann realized that only experiments can determine whether space is Euclidean.** They commissioned an experinent to look for departures from Euclidean geometry; and found none. The correct idea turned out to be to include time as well.

2. Gravitational Field Is the Metric of Space-Time

2.1. **The metric of space-time is Lorentzian.**

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

Lorentzian means that the symmetric matrix has one positive eigenvalue and three negative eigenvalues.

If $g_{\mu\nu} = \eta_{\mu\nu}$ is constant (in some co-ordinate system) there is no gravitational field. Gravitation is the effect of the departture of the metric from the Minkowski metric. That is the central idea of Einstein's theory. Even when there is a no gravitational field, $g_{\mu\nu}$might be not a constant because we are in an accelerated reference frame (curvilinear co-ordinate system). This means we feel pseudo-forces such as the centrifugal force.

2.2. **A point particle moves along a time-like geodesic of the space-time metric; light moves along a null geodesic.** Here a particle is a small object, whose mass and size is small: a planet is a particle when its orbit around the Sun is considered.

2.3. **In the Newtonian approxmation, the metric is.**

$$ds^2 \approx \left[1 + \frac{2\phi(x)}{c^2}\right] c^2 dt^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

We put back the factors of c to consider the situation where all particle velocities are small compared to that of light. The departure from Minkowski space in all except the g_{00} components is even smaller. Remember that the gravitational potential ϕ of Newton's theory has the dimensions of the square of velocity: energy per unit mass.

2.4. **If we ignore all higher order terms in $\frac{1}{c}$, the geodesic equation should reduce to Newton's equations of motion.** Let us check that this is true. The action principle gives

$$S = \frac{1}{2} \int \left[c^2 \dot{t}^2 - (\dot{x}^1)^2 - (\dot{x}^2)^{2-} (\dot{x}^3)^2\right] d\tau + \int \phi(x) \dot{t}^2 d\tau$$

The EL equations are

$$\frac{d}{dt}\left[c^2\frac{dt}{d\tau}+2\phi(x)\frac{dt}{d\tau}\right]-\frac{1}{2}\frac{\partial\phi}{\partial t}\dot{t}^2=0$$

$$-\frac{d}{dt}\left[\frac{dx^i}{d\tau}\right]-\frac{\partial\phi}{\partial x^i}\dot{t}^2=0$$

Now from the first equation

$$\frac{dt}{d\tau}\approx 1+O(c^{-1})$$

and the second equation becomes

$$\frac{d^2x^i}{dt^2}\approx-\frac{\partial\phi}{\partial x^i}.$$

Thus the newtonian potential can be thought of as changing the time component of the metric of space-time.

2.5. **Clocks tick slower in a gravitation field.** This identification has an immediate consequence: time is affected by a graviational field. A clock sitting in the field measures time evolving slower than one at infinity, far away from all sources of gravity. If a photon is emitted within the field and escapes to infinity, it will have a lower frequency than a similar one produced at infinity. This is the Gravitational Red Shift. Spectral lines of hydrogen at the surface of the Sun are red shifted compared to those at Earth: the gravitational field is much stronger on the Sun. This was one of the first predictions of General Relativity to be confirmed experimentally.

2.6. **Even the path of light is affected by gravity.** This was one of the first questions that puzzled Einstein. In Newtonian gravity, the force on a particle is proportional to its mass. The particles of light have zero mass. Are they not affected by gravity? If they are not, it can lead to a violation of the conservation of momentum: you can bounce a photon off a source of gravity. The only solution is that the path of light, null geodesics, are also affected by gravity. We can use the same Lagrangian as above, to calculate this effect, but an approach based on the eikonal equation is more convenient. We wil return to this point.