GRAVITATION F10

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Lecture 9

1. Parallel Transport

1.1. **The partial derivatives of the components of a vector do not transform as a tensor under nonlinear transformations of co-ordinates.**

$$v'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} v^{\nu}$$

By repeated use of the chain rule,

$$\frac{\partial v'^{\mu}}{\partial x'^{\rho}} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\sigma}}{\partial x'^{\rho}} \frac{\partial v^{\nu}}{\partial x^{\sigma}} + \frac{\partial^2 x'^{\mu}}{\partial x^{\nu} \partial x^{\sigma}} \frac{\partial x^{\sigma}}{\partial x'^{\rho}} v^{\nu}$$

The trouble is in the second term. Thus we need some new notion of derivative that is co-ordinate independent.

1.2. **The transport of a vector from one point to another can change its direction.** This is something we don't normally think about. To compute its derivative, we must compare the value of a vector at two neighboring points. What if the vector changes its value as it is being transported? Then we must add this to the change of the components due to their dependence on position:

$$dx^{\rho} D_{\rho} v^{\mu} = dx^{\rho} \left[\partial_{\rho} v^{\mu} + \Gamma^{\mu}_{\rho\sigma} v^{\sigma} \right]$$

We are assuming that the change of a vector due to its transport is linear in the components. The quantities $\Gamma^{\mu}_{\rho\sigma}$ must transform in some weird way

$$\Gamma^{\mu'}_{\rho\sigma} = \frac{\partial x^{\beta}}{\partial x'^{\rho}} \frac{\partial x^{\gamma}}{\partial x'^{\sigma}} \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \Gamma^{\alpha}_{\beta\gamma} + \frac{\partial^2 x'^{\mu}}{\partial x^{\beta} \partial x^{\gamma}} \frac{\partial x^{\beta}}{\partial x^{\rho'}} \frac{\partial x^{\gamma}}{\partial x^{\sigma'}}$$

to compensate for the change of the partial derivatives, so that

$$D_{\rho} v^{\mu} = \partial_{\rho} v^{\mu} + \Gamma^{\mu}_{\rho\sigma} v^{\sigma}$$

transforms as a mixed tensor. Thus Γ is not a tensor: it can be zero in one system and not be zero in another one if the transformation is nonlinear. But the offending term is symmetric under the interchange of the two lower indices. The

1.3. **The anti-symmetric part $\Gamma^{\mu}_{\rho\sigma} - \Gamma^{\mu}_{\sigma\rho}$ transforms as a tensor, called the torsion.**

1.4. **Riemannian geometry postulates that the torsion is zero.** Thus we postulate that

$$\Gamma^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\nu\mu}.$$

Furthermore

1.5. **Postulate that a scalar is unchanged under transport.** Thus the contraction with a covariant vector

$$\omega_\mu v^\mu$$

must change with the usual partial derivative

$$dx^\rho \partial_\rho \left[\omega_\mu v^\mu\right] = dx^\rho \left[D_\rho \omega_\mu\right] v^\mu + dx^\rho \omega_\mu \left[D_\rho v^\mu\right]$$

$$= dx^\rho \left[\left(D_\rho \omega_\mu\right) v^\mu + \omega_\mu \partial_\rho v^\mu + \omega_\mu \Gamma^{\mu}_{\rho\sigma} v^\sigma\right]$$

Thus

$$D_\rho \omega_\mu = \partial_\rho \omega_\mu - \Gamma^{\sigma}_{\rho\mu} \omega_\sigma$$

is the covariant derivative of a co-vector.

1.6. **Postulate that covariant derivatives satisfy the Leibnitz rule.**

$$D_\rho \left[\omega_\mu \phi_\nu\right] = \left[\partial_\rho \omega_\mu - \Gamma^{\sigma}_{\rho\mu} \omega_\sigma\right] \phi_\nu + \omega_\mu \left[\partial_\rho \phi_\nu - \Gamma^{\sigma}_{\rho\nu} \phi_\sigma\right]$$

Thus a covariant tensor $\chi_{\mu\nu}$ transforms just like a product $\omega_\mu \phi_\nu$ of two covariant vectors. If we deduce that the covariant derivative of such a tensor is

$$D_\rho \chi_{\mu\nu} = \partial_\rho \chi_{\mu\nu} - \Gamma^{\sigma}_{\rho\mu} \chi_{\sigma\nu} - \Gamma^{\sigma}_{\rho\nu} \chi_{\mu\sigma}$$

1.7. **The covariant derivative of a tensor with many indices can now be deduced.**

$$D_\rho \chi_{\mu\nu\cdots}{}^{\alpha\beta\cdots} = \partial_\rho \chi_{\mu\nu\cdots}{}^{\alpha\beta\cdots} - \Gamma^{\sigma}_{\rho\mu} \chi_{\sigma\nu\cdots}{}^{\alpha\beta\cdots} - \Gamma^{\sigma}_{\rho\nu} \chi_{\mu\sigma\cdots}{}^{\alpha\beta\cdots} + \Gamma^{\alpha}_{\rho\sigma} \chi_{\mu\nu\cdots}{}^{\sigma\beta\cdots} + \Gamma^{\beta}_{\rho\sigma} \chi_{\mu\nu\cdots}{}^{\alpha\sigma\cdots} + \cdots$$

Thus there is one term involving Γ for each index; lower indices get $-\Gamma$ and each upper index gets a term with Γ.

1.8. **Postulate that the length of a vector is unchanged under transport.** This is an essential postulate of Riemannian geometry. There are other versions of non-Euclidean geometry that do not preserve the length (e.g., Weyl geometry) . Also special geometries that require some things in addition to the length to be prepared (e.g., complex structure in Kahler geometry). We will not deal with them here. Riemannian geometry is just what we need for GR.

This amounts to assuming that the covariant derivative of the metric is zero:

$$D_\rho g_{\mu\nu} \equiv \partial_\rho g_{\mu\nu} - \Gamma^{\sigma}_{\rho\mu} g_{\sigma\nu} - \Gamma^{\sigma}_{\rho\nu} g_{\mu\sigma} = 0$$

This condition, along with the symmetry $\Gamma^{\sigma}_{\mu\nu} = \Gamma^{\sigma}_{\nu\mu}$ (zero torsion) determines Γ in terms of the derivatives of the metric components:

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left[\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right]$$

Thus,

1.9. **In Riemannian geometry, the metric determines the covariant derivative.** The Christoffel symbols appear also in the geodesic equation. From what we know now,

1.10. **The tangent vector of a geodesic is covariantly constant.** In other words, if we take the covarant derivative of the tangent vector along itself, it is zero

$$\dot{x}^\nu D_\nu \dot{x}^\mu \equiv \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0.$$

The first derivative of the curve $\dot{x}^\mu$ comes for free: we don't need to know the metric for that. But the second derivative depends on the metric. A curve whose second covariant derivative is zero is a geodesic. In General Relativity, these curves decsribe bodies that are freely falling: on which the only force acting is gravity.